Notes on the Linear Analysis of Thin-walled Beams

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CHAPTER I

SAINT-VENANT TORSION OF THIN-WALLED BEAMS

1.1. Fundamental Equations of Saint-Venant Torsion

In this section, the fundamental equations of pure torsion are derived, starting from Prandtl's assumptions about the stresses, for a prismatic beam of arbitrarily shaped cross section, made of isotropic, homogeneous material for which Hooke's law is valid. The beam is subjected to end torques T as shown in Figure 1.1. The x coordinate axis is chosen to lie along the beam axis, and the y, z coordinates in the plane of the cross section. The coordinate origin is the centroid C of one of the end sections.

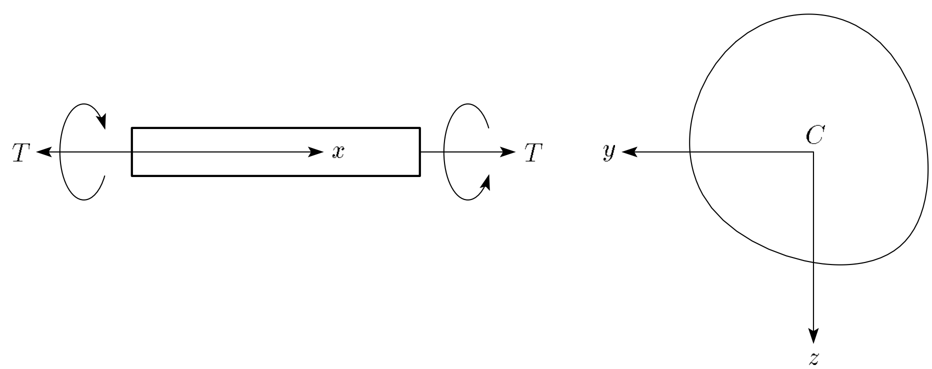

FIGURE 1.1 *Prismatic beam in torsion*

When a beam is in this state of uniform torque, it is found that only the shear stresses τ_{xy} and τ_{xz} are nonzero

$$\sigma_x = \sigma_y = \sigma_z = \tau_{yz} = 0$$

In the absence of body forces, the equations of equilibrium become

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} = 0$$

These equations show that the stresses are independent of x, which means that the shear stress distribution is the same over all cross sections.

By Hooke's law of linear elasticity, only the shear strains γ_{xy} and γ_{xz} are nonzero

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{yz} = 0$$

The nonzero shear stresses are related to the stresses by

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G}$$

where G is the shear modulus.

Only the following two of the six compatibility equations are not trivially satisfied for this state of strain

$$\frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 0$$
$$\frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{xz}}{\partial y}\right) = 0$$

In view of Hooke's law, these compatibility conditions can be written in terms of the shear stresses as

$$\frac{\partial}{\partial y}\left(-\frac{\partial \tau_{xz}}{\partial y} + \frac{\partial \tau_{xy}}{\partial z}\right) = 0$$
$$\frac{\partial}{\partial z}\left(-\frac{\partial \tau_{xz}}{\partial y} + \frac{\partial \tau_{xy}}{\partial z}\right) = 0$$

Since neither stress depends on x, the parenthesized quantity in the preceding equation is independent of x, y, and z. Consequently

$$\frac{\partial \tau_{xy}}{\partial z} - \frac{\partial \tau_{xz}}{\partial y} = -C \tag{1.1}$$

where C is a constant. This equation and the first of the equations of equilibrium form a set of two first-order partial differential equations to be solved, with the applicable boundary conditions, for the stresses.

Prandtl's stress function $\Phi(y, z)$ is defined by

$$\frac{\partial \Phi}{\partial z} = \tau_{xy} \qquad \frac{\partial \Phi}{\partial y} = -\tau_{xz}$$

Stresses calculated from the stress function Φ satisfy the equations of equilibrium, and Eq. (1.1) becomes

$$\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = -C \tag{1.2}$$

Because no forces are applied to the surface of the beam, the equality of the shear stress at the surface and the shear stress component perpendicular the boundary line of the cross section implies that this component is zero at all points of the cross-sectional boundary. Let the curvilinear coordinate s trace the boundary as shown in Figure 1.2. The s axis is tangential to the boundary in the direction of increasing s. The positive direction of the normal n to the boundary is chosen to

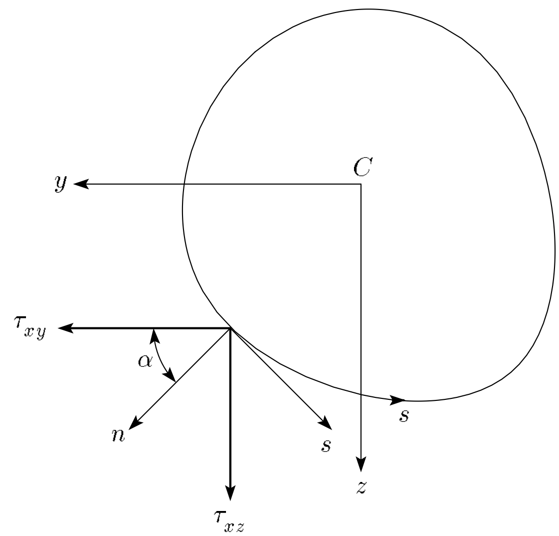

FIGURE 1.2 *Boundary condition for shear stress*

make n, s, and x a right-handed orthogonal system of axes. The boundary condition for the shear stress is

$$\tau_{xn} = \tau_{xy} \cos\alpha + \tau_{xz} \sin\alpha = 0$$

where α is the angle from the positive y axis to the positive n axis. Since

$$\frac{dy}{ds} = -\sin\alpha \qquad \frac{dz}{ds} = \cos\alpha$$

the boundary condition can be rewritten as

$$\tau_{xy}\frac{dz}{ds} - \tau_{xz}\frac{dy}{ds} = 0$$

which, in terms of the stress function, becomes

$$\frac{\partial\Phi}{\partial z}\frac{dz}{ds} + \frac{\partial\Phi}{\partial y}\frac{dy}{ds} = \frac{\partial\Phi}{\partial s} = 0$$

This shows that the value of the stress function on the boundary remains constant. When the boundary of the cross section is a single closed curve, the stress function assumes a single constant value on it, and this value may be set equal to zero. When the boundary contains several closed curves, however, an arbitrary value can be assigned to the stress function only on one of these curves. On the remaining boundary curves, the stress function assumes different values.

The stress resultants over the cross section are the two transverse shear forces V_y, V_z and the torque T, which are calculated from the shear stress distribution over the cross-sectional area A

$$V_y = \int \tau_{xy} dA = \int \frac{\partial \Phi}{\partial z} dA = 0$$
$$V_z = \int \tau_{xz} dA = -\int \frac{\partial \Phi}{\partial y} dA = 0$$
$$T = \int (y\tau_{xz} - z\tau_{xy}) dA = -\int \left(y\frac{\partial \Phi}{\partial y} + z\frac{\partial \Phi}{\partial z} \right) dA$$

The first two integrals are zero by Green's theorem, which transforms them to line integrals over the boundary curves, where Φ is constant. The third integral is evaluated as follows, by another application of Green's theorem, assuming that the boundary value of Φ has been set equal to zero

$$T = \int \left(2\Phi - \frac{\partial (y\Phi)}{\partial y} - \frac{\partial (z\Phi)}{\partial z} \right) dA = 2\int \Phi dA$$

Let u, v, and w be the displacements of a point of the cross section in the x, y, and z directions, respectively. The linear strain-displacement relations give

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0 \tag{1.3}$$

because all normal strains are zero, and the assumption of zero shear strain γ_{yz} means that

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{1.4}$$

The functional dependence of the displacements on the coordinates x, y, z determined by Eq. (1.3) is

$$u = u(y,\, z) \qquad v = v(x,\, z) \qquad w = w(x,\, y)$$

According to Eq. (1.4), the partial derivative of v with respect to z has no z dependence, and the partial derivative of w with respect to y has no y dependence. This implies that v is linear function of z and w is a linear function of y. Because the shear stresses τ_{xy} and τ_{xz} are independent of x, so are the strains γ_{xy} and γ_{xz}. The strain-displacement relations

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

imply that the dependence of v and w on x is linear.

The longitudinal displacement $u(y,\, z)$ is called the *warping displacement*. The warping displacement in Saint-Venant torsion has the same value for all cross sections. For this theory to be applicable, the beam must be unrestrained in the longitudinal direction. A cantilever beam, for instance, has a fixed end, which is not free to undergo the same warping displacement as the other sections. If external torque is applied to the free end of such a beam, normal warping stresses σ_x are developed, and Saint-Venant's solution is not applicable.

Because the in-plane shear strain γ_{yz} and all normal strains are zero, the components of the displacement in the plane of the cross section are those of a plane rigid body moving in the yz plane. It will be assumed that the axis of twist is a

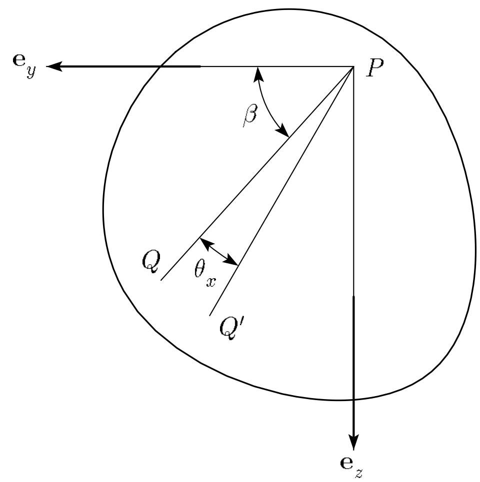

FIGURE 1.3 *Displacement of a point of the cross section*

line parallel to the beam axis and passes through the point P whose coordinates in the centroidal Cyz system are y_P and z_P. It will also be assumed that the section at $x = 0$ is restrained against rotation. The in-plane displacement of a point Q of the cross section is as shown in Figure 1.3. The point Q moves to Q' by a rotation about P

$$\mathbf{r}_{Q'P} - \mathbf{r}_{QP} = v\mathbf{e}_y + w\mathbf{e}_z$$

where $\mathbf{r}_{QP}$ denotes the position vector of Q measured from P, and $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors in the y, z directions. For small angles of twist θ_x, the displacement v is calculated as follows

$$\begin{aligned} v &= \mathbf{r}_{Q'P} \cdot \mathbf{e}_y - \mathbf{r}_{QP} \cdot \mathbf{e}_y = r\cos(\beta + \theta_x) - r\cos\theta_x \\ &= r(\cos\beta\cos\theta_x - \sin\beta\sin\theta_x - \cos\theta_x) \\ &= -r\sin\beta\,\theta_x = -(z - z_P)\theta_x \end{aligned}$$

where r is the length of $\mathbf{r}_{QP}$. A similar calculation gives w, and the displacements in the yz plane are determined to be

$$v(x,\,z) = -\theta_x(x)(z - z_P) \qquad w(x,\,y) = \theta_x(x)(y - y_P) \tag{1.5}$$

As mentioned earlier, the dependence of v and w on x is linear. Therefore

$$\theta_x(x) = Kx$$

for some constant K.

The constant C appearing in Eq. (1.1) can be evaluated in terms of the angle of twist

$$\begin{aligned} C &= -\frac{\partial \tau_{xy}}{\partial z} + \frac{\partial \tau_{xz}}{\partial y} \\ &= -G\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + G\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ &= 2G\theta'_x \end{aligned}$$

If Eq. (1.2) is solved for $C = 2G\theta'_x = 1$ and the solution is $\overline{\Phi}$, the solution Φ corresponding to a rate of twist of θ'_x is

$$\Phi = 2G\theta'_x\overline{\Phi}$$

and the torque is given by

$$T = 2\int \Phi dA = 4G\theta'_x \int \overline{\Phi} dA$$

The *torsional constant* J of the beam is defined as

$$J = 4\int \overline{\Phi} dA$$

The torsional constant, which is obtained by solving Eq. (1.2) with right hand side equal to unity and boundary conditions that depend only on the cross-sectional shape and dimensions, is a geometrical property of the cross section. The relationship between the applied torque and the angle of twist is, therefore,

$$T = GJ\theta'_x \tag{1.6}$$

1.2. Saint-Venant's Warping Function

An alternative to the stress-function approach of the preceding section is Saint-Venant's classical solution, which starts from hypotheses about the displacement field. Saint-Venant made the assumption that the cross sections rotate about the axis of twist, and even though the cross section warps out of its original plane, the projection of the deformed cross section on the yz plane retains its original shape and dimensions. The same conclusion, expressed by Eq. (1.5), was reached by the stress-function approach. If the axis of twist is taken to be the beam axis, Eq. (1.5) becomes

$$v(x,\ z) = -\theta_x(x)z \qquad w(x,\ y) = \theta_x(x)y$$

If the rate of change θ'_x of the angle of twist is assumed constant, and the end of the beam at $x = 0$ is assumed to be restrained against rotation, then Saint-Venant's displacement hypothesis about the v and w displacement components can be expressed by

$$v(x,\ z) = -\theta'_x xz \qquad w(x,\ y) = \theta'_x xy \tag{1.7}$$

The axial displacement u is assumed to be the same for all cross sections, so that it is a function of y, z only. It is also assumed that u is directly proportional to the rate of twist

$$u(y,\ z) = \theta'_x\omega(y,\ z) \tag{1.8}$$

where $\omega(y,\ z)$ is an unknown function called the *warping function*.

The strain-displacement relations determine the strains from the assumed displacement field

$$\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = 0 \\
\epsilon_y &= \frac{\partial v}{\partial y} = 0 \\
\epsilon_z &= \frac{\partial w}{\partial z} = 0 \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\theta_x' x + \theta_x' x = 0 \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \theta_x' \left(\frac{\partial \omega}{\partial y} - z \right) \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \theta_x' \left(\frac{\partial \omega}{\partial z} + y \right)
\end{aligned}$$

The stresses are then given by Hooke's law

$$\begin{aligned}
\sigma_x &= 0 \\
\sigma_y &= 0 \\
\sigma_z &= 0 \\
\tau_{yz} &= 0 \\
\tau_{xy} &= G\theta_x' \left(\frac{\partial \omega}{\partial y} - z \right) \\
\tau_{xz} &= G\theta_x' \left(\frac{\partial \omega}{\partial z} + y \right)
\end{aligned}$$

The ratio of the change in volume to the original volume, called the *cubical dilatation*, is zero

$$e = \epsilon_x + \epsilon_y + \epsilon_z = 0$$

and all surface forces are zero, so that the displacement formulation of the equations of elasticity reduces to

$$\nabla^2 u = \nabla^2 v = \nabla^2 w = 0$$

where ∇^2 is the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The partial differential equations for the displacement components v and w are trivially satisfied. The equation for the warping displacement u gives

$$\nabla^2 \omega = \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = 0 \tag{1.9}$$

The boundary condition for the shear stresses on the cylindrical surface of the beam, shown in Figure 1.2, is

$$\tau_{xy} \cos\alpha + \tau_{xz} \sin\alpha = 0$$

which, in terms of the warping function, becomes

$$\left(\frac{\partial\omega}{\partial y} - z\right)\cos\alpha + \left(\frac{\partial\omega}{\partial z} + y\right)\sin\alpha = 0 \tag{1.10}$$

The torque at any section is

$$T = \int (y\tau_{xz} - z\tau_{xy})dA = G\theta_x' \int \left[\left(\frac{\partial\omega}{\partial z} + y\right)y - \left(\frac{\partial\omega}{\partial y} - z\right)z\right]dA$$

The integral in the preceding equation is identified as the torsional constant J

$$J = I_y + I_z + \int \left(y\frac{\partial\omega}{\partial z} - z\frac{\partial\omega}{\partial y}\right)dA$$

The area integral can be transformed into a line integral over the boundary by applying Green's theorem

$$\int \left(\frac{\partial(y\omega)}{\partial z} - \frac{\partial(z\omega)}{\partial y}\right)dA = -\oint \omega(zdz + ydy) = -\oint \omega\mathbf{r}\cdot d\mathbf{r}$$

where $\mathbf{r}$ denotes the position vector from the centroid to points on the boundary of the cross section. If the cross section is multiply connected, then the boundary integral is the sum of the line integrals along individual parts of the boundary. The torsional constant is given by

$$J = I_y + I_z - \oint \omega\mathbf{r}\cdot d\mathbf{r} \tag{1.11}$$

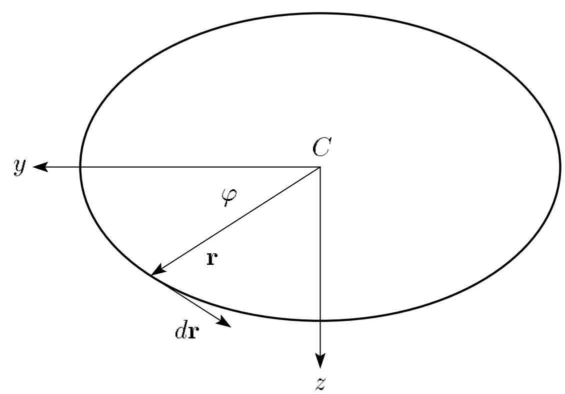

FIGURE 1.4 *Solid elliptic cross section*

Closed-form solutions for the warping function ω are known only for simple and regular geometric shapes, such as the solid elliptic cross section shown in Figure 1.4. The equation of the elliptical boundary is

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

The warping function is known from the theory of elasticity

$$\omega = -\frac{a^2 - b^2}{a^2 + b^2}yz$$

The line integral in Eq. (1.11) can be evaluated with the parametric representation of the ellipse in terms of the angle φ

$$\begin{aligned}\oint \omega \mathbf{r} \cdot d\mathbf{r} &= -\frac{a^2 - b^2}{a^2 + b^2} \oint yz(ydy + zdz) \\ &= -\frac{a^2 - b^2}{a^2 + b^2} \int_0^{2\pi} (ba \cos^2 \varphi \sin^2 \varphi)(-a^2 + b^2) d\varphi \\ &= \frac{ab(b^2 - a^2)^2 \pi}{4(a^2 + b^2)}\end{aligned}$$

The area moments of inertia are

$$I_y = \frac{1}{4}\pi ab^3 \qquad I_z = \frac{1}{4}\pi ba^3$$

and the torsional constant is

$$J = I_y + I_z - \frac{ab(b^2 - a^2)^2 \pi}{4(a^2 + b^2)} = \frac{\pi a^3 b^3}{a^2 + b^2}$$

1.3. Thin-walled Open Sections

A thin-walled section is called *open* if the centerline of its walls is not a closed curve. Equivalently, a thin-walled section whose boundary is a single piecewise continuous closed curve is an open section. The simplest open thin-walled section is the narrow rectangular strip shown in Figure 1.5, for which the wall thickness t is less than one-tenth of the length h. An approximate value for the torsional

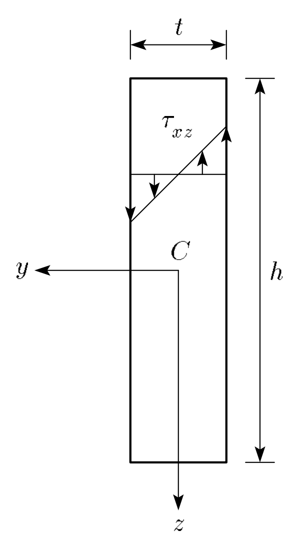

FIGURE 1.5 *Narrow rectangular cross section*

constant J for this section will be obtained by assuming that τ_{xy} is negligibly small and the shear stress τ_{xz} varies linearly across the wall thickness

$$\tau_{xy} = 0 \qquad \tau_{xz} = \frac{2\tau_{\max} y}{t}$$

The equation to be solved is Eq. (1.1), which, with these assumptions, becomes

$$2G\theta'_x = C = \frac{\partial \tau_{xz}}{\partial y} = \frac{2\tau_{\max}}{t}$$

This determines the differential equation for the stress function

$$-\frac{d\Phi}{dy} = \tau_{xz} = 2G\theta'_x y$$

When this equation is integrated and the stress function is set equal to zero on the longer edges of the rectangle, the result is

$$\Phi(y) = G\theta'_x \left(\frac{t^2}{4} - y^2\right)$$

and the torsional constant is found to be

$$J = 4\int \overline{\Phi} dA = 2\int \left(\frac{t^2}{4} - y^2\right) dA = \frac{t^2 A}{2} - 2I_z = \frac{t^3 h}{2} - 2\frac{t^3 h}{12} = \frac{t^3 h}{3}$$

where A is the area and I_z is the area moment of inertia about the z axis.

In this solution, it is not possible to set the value of the stress function to zero on the shorter edges of the rectangle. Consequently, the stress distribution

$$\tau_{xz} = \frac{2Ty}{J}$$

is not valid near the shorter edges, where the boundary conditions require that the shear stress be zero. In addition, the torque due to τ_{xz} is one-half the actual torque T. This is partially because the neglected shear stresses τ_{xy} are concentrated near the shorter edges and have longer moment arms than the stresses τ_{xz}.

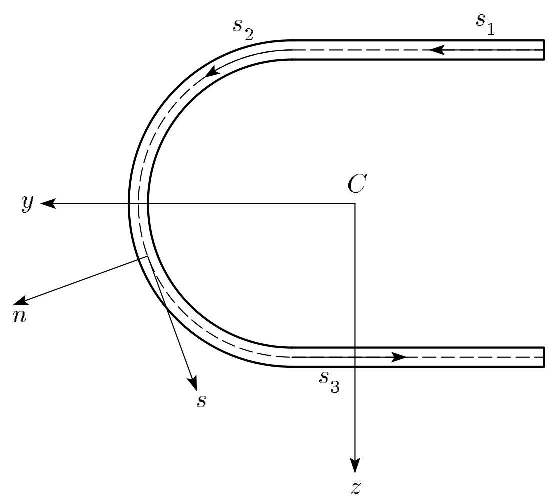

FIGURE 1.6 *Horseshoe section*

The approximate results obtained for a narrow rectangular strip can be applied to more complicated thin-walled open sections, such as the horseshoe section shown in Figure 1.6. Saint-Venant's approximation for the torsional constant is

$$J = \frac{1}{3}\int t^3(s)ds \tag{1.12}$$

where s is the coordinate that traces the median line of the section and $t(s)$ is the wall thickness. The shear stress distribution is

$$\tau_{xz} = \frac{2Tn}{J} \tag{1.13}$$

where n is the normal coordinate measured from the median line. The maximum shear stress occurs at the maximum wall thickness $t_{\max}$

$$\tau_{\max} = \frac{Tt_{\max}}{J} \tag{1.14}$$

1.4. Thin-walled Closed Sections

A thin-walled section is called *closed* if the centerline of its walls is a closed curve. Equivalently, a thin-walled section whose boundary is formed by two piecewise continuous closed curves is a closed section. The boundary of a *multicell* closed section is made up of more than two closed curves.

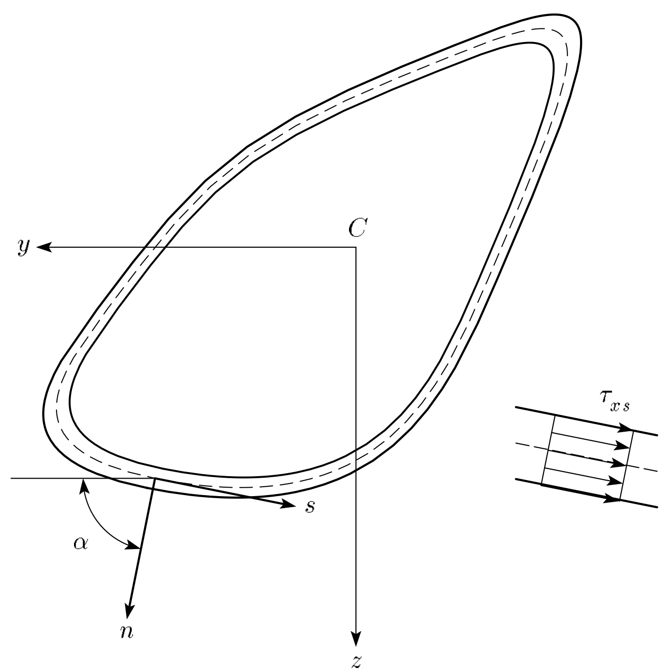

FIGURE 1.7 *Closed thin-walled section*

A closed thin-walled cross section is shown in Figure 1.7. The tangential and normal coordinates, s and n, are chosen so that the axes n, s, x form a right-handed triad. The coordinate s traces the median line starting from an arbitrarily selected origin, and the y, z coordinates of any point on the median line are functions of s.

The normal coordinate n of any point of the median line is zero. The angle $\alpha(s)$ is measured from the positive y axis to the positive n axis. It will be assumed that the shear stress is tangent to the median line and does not vary across the wall thickness. The *shear flow* q due to the shear stress τ_{xs}, defined by,

$$q = t(s)\tau_{xs}(s)$$

will be assumed constant. Then, at any s, the derivative of the stress function with respect to n is

$$\begin{aligned}\frac{\partial \Phi}{\partial n} &= \frac{\partial \Phi}{\partial y}\frac{dy}{dn} + \frac{\partial \Phi}{\partial z}\frac{dz}{dn} = -\tau_{xz}\cos\alpha + \tau_{xy}\sin\alpha \\ &= -\tau_{xs} = -\frac{q}{t(s)}\end{aligned}$$

The stress function is then determined by setting its value equal to zero on the outer boundary of the section

$$\Phi(n,s) = \frac{q}{2}\left(1 - \frac{2n}{t(s)}\right)$$

The derivatives of the axial displacement are

$$\begin{aligned}\frac{\partial u}{\partial y} &= \gamma_{xy} - \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} + \theta'_x(z - z_P) \\ \frac{\partial u}{\partial z} &= \gamma_{xz} - \frac{\partial w}{\partial x} = \frac{\tau_{xz}}{G} - \theta'_x(y - y_P)\end{aligned}$$

where P is a point on the axis of twist. Thus, on the median line, the derivative of u with respect to s is

$$\frac{\partial u}{\partial s} = \frac{\tau_{xs}}{G} + \theta'_x(z - z_P)\frac{dy}{ds} - \theta'_x(y - y_P)\frac{dz}{ds}$$

Let $\mathbf{r}_P(s)$ denote the position vector of the point at s measured from the point P

$$\mathbf{r}_P = (y - y_P)\mathbf{e}_y + (z - z_P)\mathbf{e}_z$$

where $\mathbf{e}_y$, $\mathbf{e}_z$ are unit vectors in the positive y, z directions, respectively. The unit tangent vector $\mathbf{e}_s$ at the point with coordinates y, z is

$$\mathbf{e}_s = \frac{\partial y}{\partial s}\mathbf{e}_y + \frac{\partial z}{\partial s}\mathbf{e}_z$$

The unit normal vector $\mathbf{e}_n$ is, therefore, given by

$$\mathbf{e}_n = \mathbf{e}_s \times \mathbf{e}_x = \frac{\partial z}{\partial s}\mathbf{e}_y - \frac{\partial y}{\partial s}\mathbf{e}_z$$

The projection of the position vector $\mathbf{r}_P$ onto the unit normal vector is

$$\mathbf{r}_P \cdot \mathbf{e}_n = (y - y_P)\frac{\partial z}{\partial s} - (z - z_P)\frac{\partial y}{\partial s}$$

The derivative of the warping displacement with respect to s becomes

$$\frac{\partial u}{\partial s} = \frac{\tau_{xs}}{G} - \theta'_x \mathbf{r}_P \cdot \mathbf{e}_n \tag{1.15}$$

The line integral of this derivative over the closed path formed by the median line of the cross section is zero

$$\frac{1}{G}\oint \tau_{xs}\,ds - \theta_x' \oint \mathbf{r}_P \cdot \mathbf{e}_n\,ds = 0 \tag{1.16}$$

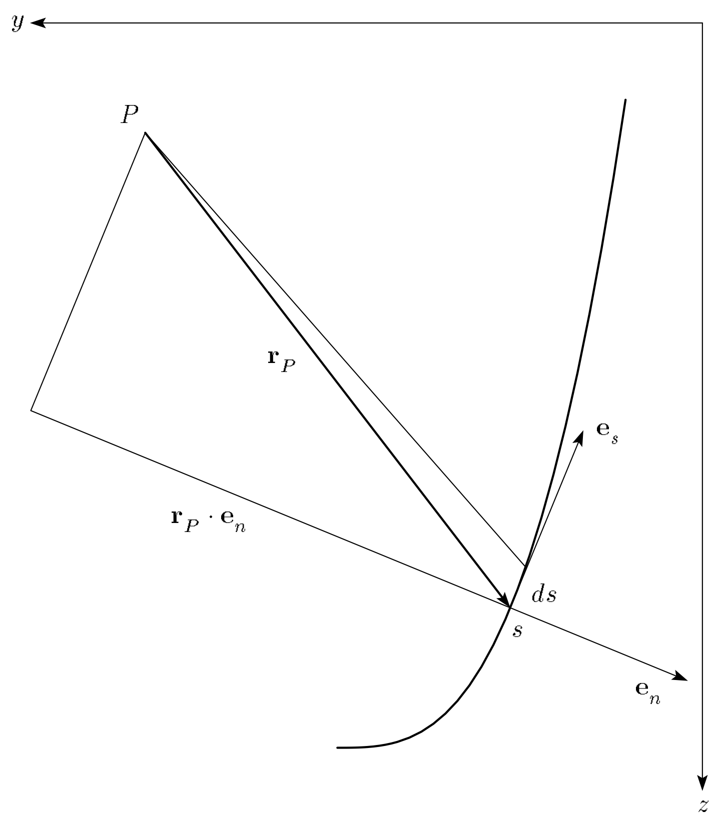

FIGURE 1.8 *Definition of sectorial area*

In the preceding equation, the integral

$$\Omega = \oint \mathbf{r}_P \cdot \mathbf{e}_n\,ds$$

is interpreted as twice the area enclosed by the median line. Figure 1.8 shows that the differential quantity under the integral is twice the area of the triangle with base length ds. As the position vector sweeps through the entire median line, the integral gives the twice the area enclosed by the median line. In terms of the constant shear flow q, Eq. (1.16) is rewritten as

$$\frac{q}{G}\oint \frac{ds}{t(s)} - \theta_x'\Omega = 0$$

The torque resultant is calculated as the moment due to the shear stress about the point P

$$\begin{aligned} T &= \mathbf{e}_x \cdot \int \mathbf{r}_P \times \tau_{xs} t ds \mathbf{e}_s = q \int \mathbf{r}_P \cdot (\mathbf{e}_s \times \mathbf{e}_x) ds \\ &= q \int \mathbf{r}_P \cdot \mathbf{e}_n ds = q\Omega = \frac{G\theta'_x \Omega^2}{\oint \frac{ds}{t(s)}} \end{aligned}$$

This equation gives the shear stress

$$\tau_{xs}(s) = \frac{T}{t(s)\Omega} \tag{1.17}$$

and the torsional constant

$$J = \frac{T}{G\theta'_x} = \frac{\Omega^2}{\oint \frac{ds}{t(s)}} \tag{1.18}$$

For constant wall thickness, the torsional constant becomes

$$J = \frac{\Omega^2 t}{S}$$

where S denotes the length of the median line.

CHAPTER II

THIN-WALLED ELASTIC BEAMS OF OPEN CROSS SECTION

2.1. Geometry of Deformation

Figure 2.1 shows a prismatic thin-walled beam and its cross section. The beam axis, which is defined as the line of centroids of the cross sections, is chosen to lie along the x axis. Points on a particular cross section are specified by defining their y and z coordinates. The coordinate s traces the median line of the cross section. Each value of s corresponds to a well-defined point of the median line, so that the coordinates y and z of a point on the median line are functions of s.

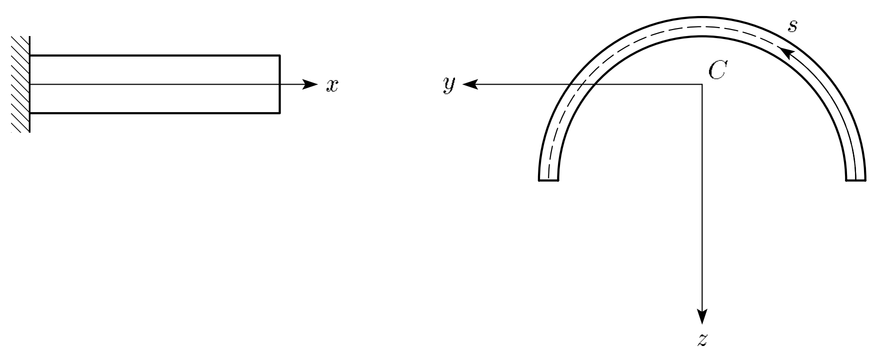

FIGURE 2.1 *A thin-walled beam and its cross section*

It will be assumed that the shape of the median line and its dimensions remain unchanged in the yz plane when the beam undergoes a deformation under static loads. This means that the transverse displacements, which are defined as the displacement components in the plane of the undeformed cross section, of a point on the median line are those of a point belonging to a plane rigid curve constrained to move in its own plane. Let A and B be arbitrarily chosen points of such a plane rigid body in its initial position. After the body undergoes a displacement, the points A and B occupy new positions in space. Let A', B' be the projections of these new positions onto the yz plane, as shown in Figure 2.2. Let $\mathbf{r}_{BA}$ be the position vector of point B measured from point A. The vector $\mathbf{r}_{B'A'}$ is given by

$$\mathbf{r}_{B'A'} = \mathbf{r}_{BA} \cos\theta_x + \mathbf{e}_x \times \mathbf{r}_{BA} \sin\theta_x \tag{2.1}$$

where $\mathbf{e}_x$ is the unit vector in the direction of the positive x axis, and θ_x is the angle measured from the vector $\mathbf{r}_{AB}$ to the vector $\mathbf{r}_{A'B'}$, $-\pi \leqslant \theta_x \leqslant \pi$, with the vector

$\mathbf{r}_{AB}$ translated such that the points A and A' become coincident. The rotation is counterclockwise, if the cross product $\mathbf{r}_{BA} \times \mathbf{r}_{B'A'}$, evaluated according to the right-hand rule, has the same direction as $\mathbf{e}_x$. Since

$$\mathbf{r}_{BA} \times \mathbf{r}_{B'A'} = \mathbf{r}_{BA} \times (\mathbf{e}_x \times \mathbf{r}_{BA}) \sin\theta_x = |\mathbf{r}_{BA}|^2 \sin\theta_x \mathbf{e}_x$$

the rotation is counterclockwise for $\sin\theta_x \geqslant 0$. This establishes that the sign of θ_x is positive when the sense of rotation from AB to $A'B'$ is counterclockwise. For small rotations, Eq. (2.1) becomes

$$\mathbf{r}_{B'A'} = \mathbf{r}_{BA} + \theta_x \mathbf{e}_x \times \mathbf{r}_{BA} \tag{2.2}$$

Let v_A and w_A be the displacement components of point A along the y and z axes, and let v_B and w_B be the corresponding displacement components of point B. Let the transverse displacement vector be denoted by $\mathbf{u}_A$ for point A and by $\mathbf{u}_B$ for point B

$$\mathbf{u}_A = \mathbf{r}_{A'A} \qquad \mathbf{u}_B = \mathbf{r}_{B'B}$$

From the vector polygon $A'ABB'$ in Figure 2.2

$$\mathbf{r}_{B'A'} - \mathbf{r}_{BA} = \mathbf{r}_{B'B} - \mathbf{r}_{A'A} = \mathbf{u}_B - \mathbf{u}_A = (v_B - v_A)\mathbf{e}_y + (w_B - w_A)\mathbf{e}_z$$

and Eq. (2.2) can be written in terms of displacement as

$$\mathbf{u}_B = \mathbf{u}_A + \theta_x \mathbf{e}_x \times \mathbf{r}_{BA} \tag{2.3}$$

The scalar components of this equation are

$$\begin{aligned} v_B &= v_A + (z_B - z_A)\theta_x \\ w_B &= w_A - (y_B - y_A)\theta_x \end{aligned} \tag{2.4}$$

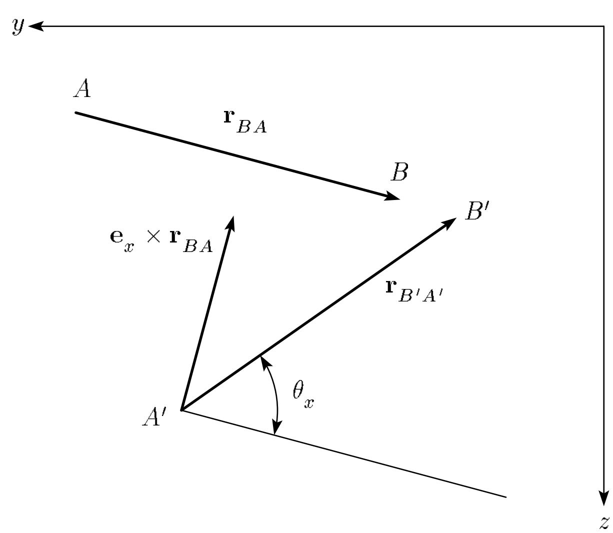

FIGURE 2.2 *Geometry of plane rigid body motion*

To apply the foregoing considerations to a thin-walled beam, let A be an arbitrarily chosen reference point, which need not be a material point of the median line, but which is assumed to be displaced as if it were rigidly attached to the median line. Let $\mathbf{e}_s$ be a unit vector tangent to the median line in the direction of inreasing s as shown in Figure 2.3. The unit normal vector $\mathbf{e}_n$ is defined so as to make the triad $\mathbf{e}_n$, $\mathbf{e}_s$, and $\mathbf{e}_x$ a right-handed set of orthogonal vectors

$$\mathbf{e}_x = \mathbf{e}_n \times \mathbf{e}_s \qquad \mathbf{e}_n = \mathbf{e}_s \times \mathbf{e}_x$$

Let $\eta(s)$ denote the tangential component of the displacement of the point of the median line at the coordinate s. This component is given by Eq. (2.3)

$$\eta(s) = \mathbf{u}_A \cdot \mathbf{e}_s + \theta_x \mathbf{e}_s \cdot (\mathbf{e}_x \times \mathbf{r}_A(s)) = \mathbf{u}_A \cdot \mathbf{e}_s + \theta_x (\mathbf{e}_s \times \mathbf{e}_x) \cdot \mathbf{r}_A(s)$$

where $\mathbf{r}_A(s)$ is the position vector of the point at s measured from A. In terms of the angle β between the s and y axes, this equation becomes

$$\eta(s) = v_A \cos\beta + w_A \sin\beta + \theta_x \mathbf{r}_A \cdot \mathbf{e}_n$$

Let r_A^n denote the projection of the vector $\mathbf{r}_A$ onto the unit normal

$$r_A^n = \mathbf{r}_A \cdot \mathbf{e}_n$$

The tangential component of the displacement of the point at s is given by

$$\eta(x,\, s) = v_A(x) \cos\beta(s) + w_A(x) \sin\beta(s) + \theta_x(x) r_A^n(s) \tag{2.5}$$

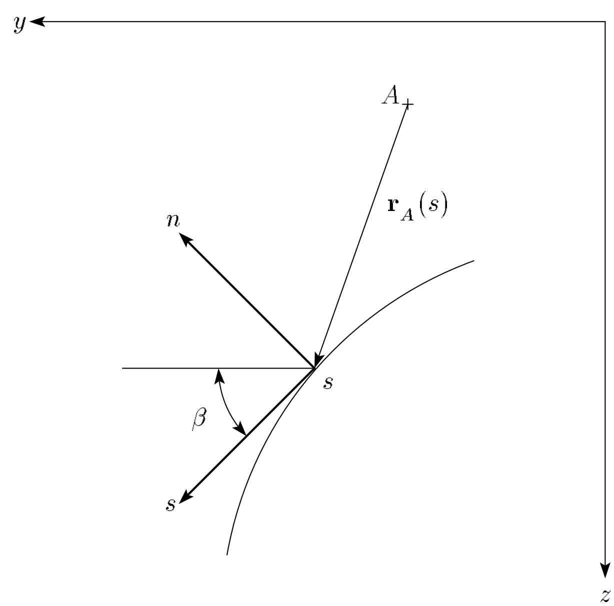

FIGURE 2.3 *Tangential and normal components of displacement*

It will now be assumed that the shear strain γ_{xs} is negligible in a thin-walled beam with open cross section. This means that longitudinal fibers of the beam material remain orthogonal to the fibers along the median line. This assumption can be written as

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \eta}{\partial x} = 0$$

where u is the displacement of the point at s along the x axis. This leads to

$$\frac{\partial u}{\partial s} = -\frac{\partial \eta}{\partial x} = -v_A'(x)\cos\beta(s) - w_A'(x)\sin\beta(s) - \theta_x'(x) r_A^n(s)$$

in which prime denotes differentiation with respect to x. Integration with respect to s gives

$$\begin{aligned} u(x,\, s) &= -v_A'(x)\int_0^s \cos\beta(s)ds - w_A'(x)\int_0^s \sin\beta(s)ds - \theta_x'(x)\int_0^s r_A^n(s)ds \\ &= -v_A'(x)y(s) - w_A'(x)z(s) - \theta_x'(x)\omega_A(s) + u_0(x) \end{aligned}$$

where

$$\omega_A(s) = \int_0^s r_A^n(s)ds = \int_0^s \mathbf{r}_A(s)\cdot \mathbf{e}_n ds$$

is the *sectorial area* and $u_0(x)$ is the longitudinal displacement of the point of the median line at $s = 0$.

The longitudinal strain ϵ is calculated by differentiating the longitudinal displacement with respect to x

$$\frac{\partial u}{\partial x} = \epsilon_x(x,\, s) = u_0'(x) - v_A''(x)y(s) - w_A''(x)z(s) - \theta_x''(x)\omega_A(s) \tag{2.6}$$

The first three terms of this equation are consistent with the Navier-Bernoulli hypothesis that plane sections remain plane. The contribution of the warping of the section is expressed by the last term. For this reason the sectorial area ω_A is called the *warping function*. The warping function depends on the *sectorial origin*, which is the origin chosen for the coordinate s, and on the reference point A, termed the *pole* of the warping function.

2.2. Properties of the Warping Function

The warping function ω_A, with pole at point A and origin at $s = s_0$, is defined as the integral

$$\omega_A(s) = \int_{s_0}^s \mathbf{r}_A(s)\cdot \mathbf{e}_n(s)ds$$

where $\mathbf{r}_A(s)$ is the position vector of the point at s of the median line measured from the pole A, as shown in Figure 2.4. The direction of $\mathbf{e}_n(s)$ is determined from the convention that the axes $(n,\, s,\, x)$ are right-handed, with the s axis in the direction of increasing s. The magnitude of the differential quantity $\mathbf{r}_A(s)\cdot\mathbf{e}_n(s)ds$ is twice the area of the triangle with base ds and height $\mathbf{r}_A(s)\cdot\mathbf{e}_n(s)$. The sign of this quantity is positive if the projection of the position vector onto the unit normal vector $\mathbf{e}_n(s)$ is positive. This sign is more conveniently determined on the basis of the sense of rotation of the position vector as it sweeps through the area in the direction of increasing s. If this rotation is clockwise, the contribution to the integral is negative, and if it is counterclockwise, the contribution is positive. This is verified for any point of the median line by writing the projection of the position vector onto the unit normal in the form

$$\mathbf{r}_A\cdot\mathbf{e}_n = \mathbf{r}_A\cdot(\mathbf{e}_s\times\mathbf{e}_x) = (\mathbf{r}_A\times\mathbf{e}_s)\cdot\mathbf{e}_x$$

Thus, when the rotation of the vector $\mathbf{r}_A$ is counterclockwise as its tip moves in the direction of $\mathbf{e}_s$, the cross product $\mathbf{r}_A\times\mathbf{e}_s$ is in the same direction as $\mathbf{e}_x$, making $\mathbf{r}_A\cdot\mathbf{e}_n$ positive.

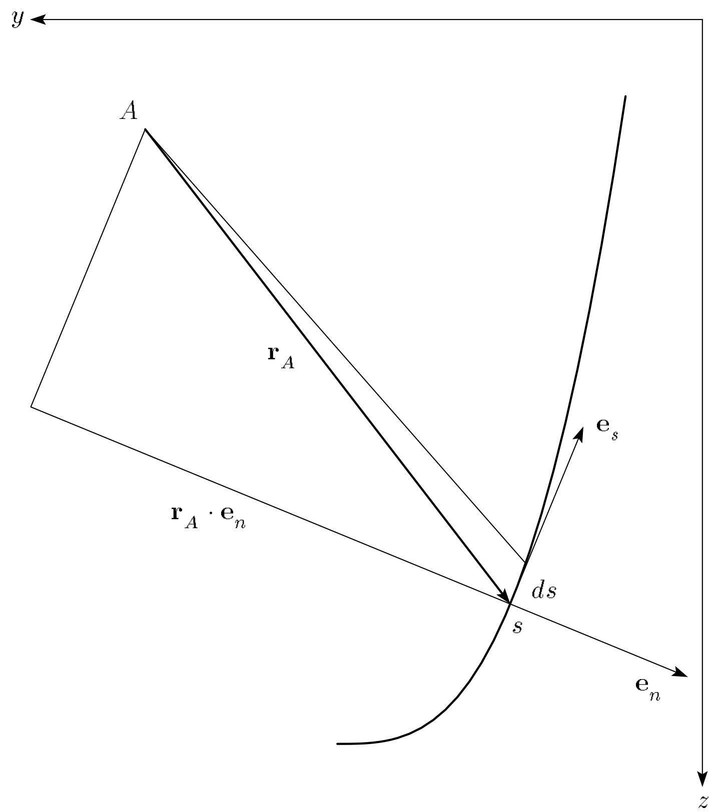

FIGURE 2.4 *Definition of sectorial area*

Let A and B be two arbitrarily selected poles for the warping function. Suppose that the origin for ω_A is chosen to be at $s = s_0$ and the origin for ω_B at $s = s_1$, as shown in Figure 2.5. The relationship between ω_A and ω_B is found by the computation

$$
\begin{aligned}
\omega_A(s) &= \int_{s_0}^{s} \mathbf{r}_B \cdot \mathbf{e}_n ds = \int_{s_0}^{s} (\mathbf{r}_B + \mathbf{r}_{BA}) \cdot \mathbf{e}_n ds \\
&= \int_{s_0}^{s_1} \mathbf{r}_B \cdot \mathbf{e}_n ds + \int_{s_1}^{s} \mathbf{r}_B \cdot \mathbf{e}_n ds + \int_{s_0}^{s} \mathbf{r}_{BA} \cdot \mathbf{e}_n ds \\
&= -\omega_B(s_0) + \omega_B(s) + \int_{s_0}^{s} \mathbf{r}_{BA} \cdot (\sin\beta \mathbf{e}_y - \cos\beta \mathbf{e}_z) ds \\
&= -\omega_B(s_0) + \omega_B(s) + \int_{s_0}^{s} \mathbf{r}_{BA} \cdot (dz \mathbf{e}_y - dy \mathbf{e}_z) \\
&= -\omega_B(s_0) + \omega_B(s) + (y_B - y_A)(z(s) - z_0) - (z_B - z_A)(y(s) - y_0)
\end{aligned}
$$

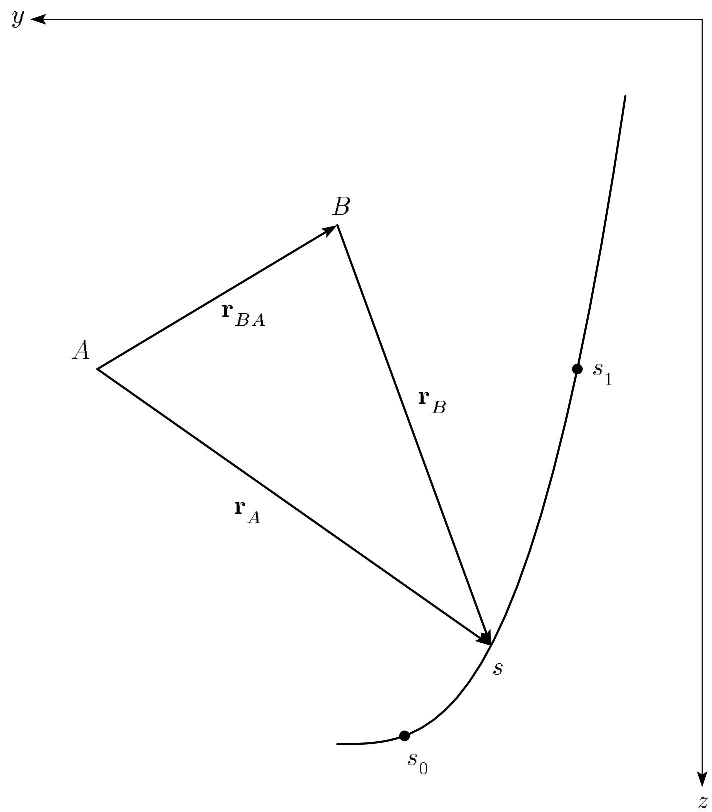

FIGURE 2.5 *Poles and origins of warping functions*

In this calculation, β is the angle between the s and y axes as shown in Figure 2.3, and y_0, z_0 are the coordinates of the origin chosen for ω_A

$$y_0 = y(s_0) \qquad z_0 = z(s_0)$$

The equation for finding the warping function ω_A with origin s_0 from the the warping function ω_B with origin s_1 is, therefore,

$$\omega_A(s) = \omega_B(s) - \omega_B(s_0) + (z_A - z_B)(y(s) - y_0) - (y_A - y_B)(z(s) - z_0) \qquad (2.7)$$

When A and B are coincident points, but s_0 and s_1 are two distinct origins, the transformation equation becomes

$$\omega_A(s) = \omega_B(s) - \omega_B(s_0) \qquad (2.8)$$

showing that the effect of changing the origin of a warping function without changing its pole is to add a constant to it. If, on the other hand, the origins s_0, s_1 are the same and the poles A, B are different, the transformation equation is

$$\omega_A(s) = \omega_B(s) + (z_A - z_B)(y(s) - y_0) - (y_A - y_B)(z(s) - z_0) \qquad (2.9)$$

since, in this case,

$$\omega_B(s_0) = \omega_B(s_1) = 0$$

If $\omega(s)$ is a warping function for a particular pole and origin, the area integral

$$Q_\omega = \int \omega(s) dA$$

is called the *first sectorial moment*. The area integrals

$$I_{y\omega} = \int y(s)\omega(s) dA$$
$$I_{z\omega} = \int z(s)\omega(s) dA$$

are known as the *sectorial products* of area. These definitions are analogous to the definitions of the first, second, and product moments of area

$$Q_y = \int z dA$$
$$Q_z = \int y dA$$
$$I_y = \int z^2 dA$$
$$I_z = \int y^2 dA$$
$$I_{yz} = \int yz dA$$

A pole for which the the sectorial products of area are both zero is called a *principal pole*. Let A and B be two poles for the warping function with origins at s_0 and s_1, respectively. By multiplying both sides of Eq. (2.7) by y and integrating both sides of the result over the cross sectional area, one obtains

$$I_{y\omega_A} = I_{y\omega_B} - \omega_B(s_0)Q_z + (z_A - z_B)(I_z - y_0 Q_z) - (y_A - y_B)(I_{yz} - z_0 Q_z)$$

Since the origin of the y, z axes is the centroid C of the cross section, the first moments Q_y and Q_z are both zero, so that

$$I_{y\omega_A} = I_{y\omega_B} + (z_A - z_B)I_z - (y_A - y_B)I_{yz} \tag{2.10}$$

A similar calculation gives

$$I_{z\omega_A} = I_{z\omega_B} + (z_A - z_B)I_{yz} - (y_A - y_B)I_y \tag{2.11}$$

The conditions for A to be a principal pole

$$I_{y\omega_A} = I_{z\omega_A} = 0$$

are solved for the coordinates of the pole

$$y_A = y_B + \frac{I_{z\omega_B} I_z - I_{y\omega_B} I_{yz}}{I_y I_z - I_{yz}^2} \tag{2.12}$$

$$z_A = z_B + \frac{I_{z\omega_B} I_{yz} - I_{y\omega_B} I_y}{I_y I_z - I_{yz}^2} \tag{2.13}$$

These expressions do not depend on the origin chosen for the pole B, because if this origin is shifted, the resulting warping function $\overline{\omega}_B$ differs from ω_B by a constant value, say K, and

$$I_{y\overline{\omega}_B} = \int y(\omega_B + K)dA = I_{y\omega_B} + KQ_z = I_{y\omega_B}$$

Hence, the sectorial products of area for the warping function ω_B remain the same when the sectorial origin is changed, and the coordinates determined by Eqs. (2.12) and (2.13) are independent of this origin.

It is important to realize that the coordinates given by Eqs. (2.12) and (2.13) are also independent of the pole B. If D is any arbitary pole, its sectorial products of area are related to those of the pole B by

$$\begin{aligned} I_{y\omega_B} &= I_{y\omega_D} + (z_B - z_D)I_z - (y_B - y_D)I_{yz} \\ I_{z\omega_B} &= I_{z\omega_D} + (z_B - z_D)I_{yz} - (y_B - y_D)I_y \end{aligned}$$

Then

$$I_{z\omega_B}I_z - I_{y\omega_B}I_{yz} = I_{z\omega_D}I_z - I_{y\omega_D}I_{yz} - (y_B - y_D)(I_yI_z - I_{yz}^2)$$

which, when substituted into Eq. (2.12), gives

$$y_A = y_D + \frac{I_{z\omega_D}I_z - I_{y\omega_D}I_{yz}}{I_yI_z - I_{yz}^2}$$

The right side of this equation is the expression for the y coordinate of A with the pole D. Similarly, the z coordinate of A remains the same regardless of the pole used to find it. Thus, the principal pole depends only on the cross-sectional shape and dimensions; it is a cross-sectional property.

If, for a given pole A, there is a sectorial origin s_0 such that

$$Q_{\omega_A} = \int \omega_A(s)dA = 0$$

the point s_0 is termed a *principal origin*. To determine s_0, let B be a pole coincident with A but with a known origin s_1. According to Eq. (2.8)

$$\omega_A(s) = \omega_B(s) - \omega_B(s_0)$$

so that the condition for s_0 to be a principal origin is

$$Q_{\omega_A} = Q_{\omega_B} - \omega_B(s_0)A = 0 \tag{2.14}$$

This equation determines the principal origin s_0 in terms of the arbitrarily selected origin s_1. The existence of at least one s_0 is ensured, for most cross sectional shapes, by the mean value theorem for integrals. There may be multiple solutions for s_0, in which case any one solution can be selected as the principal origin. If another pole D, which is coincident with A and B but has its origin at s_2, is used instead of B in determining the principal origin s_0, then

$$Q_{\omega_D} = Q_{\omega_B} - \omega_B(s_2)A$$

and the condition for s_0 to be a principal origin, written in terms of D, becomes

$$\begin{aligned} Q_{\omega_A} &= Q_{\omega_D} - \omega_D(s_0)A = Q_{\omega_B} - \omega_B(s_2)A - (\omega_B(s_0) - \omega_B(s_2))A \\ &= Q_{\omega_B} - \omega_B(s_0)A = 0 \end{aligned}$$

This shows that the same principal origin is obtained regardless of where the reference origin s_1 is placed. Hence, the principal origin is a cross-sectional property.

Let A be the principal pole for the warping function ω_A, whose origin has been selected arbitrarily. When this origin is changed to a principal origin, the sectorial products of area for the warping function remain zero, because, as mentioned above, these products are independent of the sectorial origin as long as the centroid is used as the origin of the coordinates y and z. Hence, for a given cross section, it is possible to find a pole A and an origin s_0 such that Q_{ω_A}, $I_{y\omega_A}$, and $I_{z\omega_A}$ are zero. A warping function satisfying these conditions is termed a *principal* warping function. Principal warping functions will henceforth be written without a subscript.

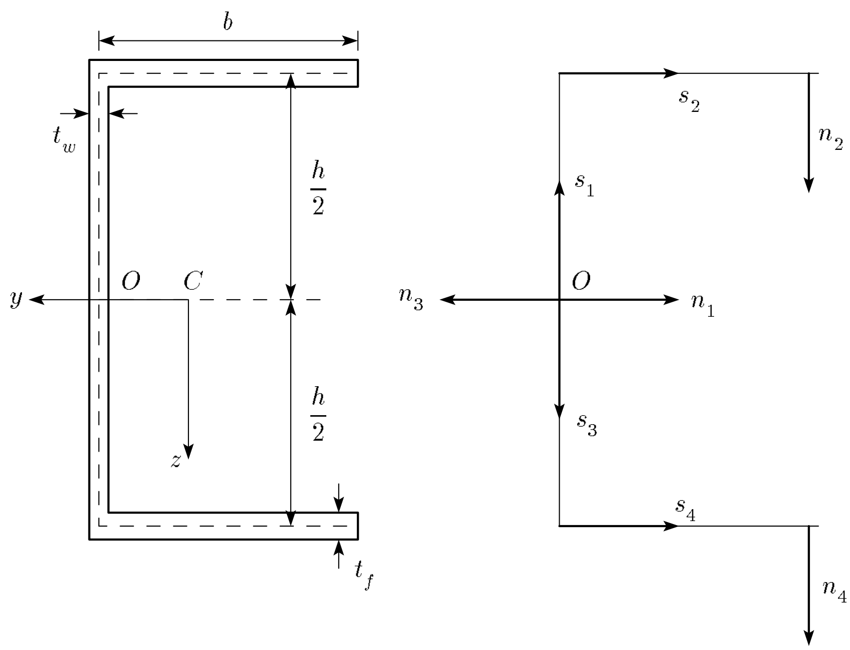

FIGURE 2.6 *Symmetric channel section*

For the symmetric channel section shown in Figure 2.6, the warping function with pole and origin both at the point of intersection O of the y axis and the median line is given by

$$\omega_O(s_1) = 0$$

$$\omega_O(s_2) = -\frac{h}{2}s_2$$

$$\omega_O(s_3) = 0$$

$$\omega_O(s_4) = \frac{h}{2}s_4$$

where the signs are determined from the sense of rotation of the vector from O to points on the median line. For instance, $\omega_O(s_2)$ is negative, because the position vector rotates clockwise as it traces the median line of the upper flange.

Let A be the principal pole of the cross section shown in Figure 2.6. The coordinates of A will be found by using O both as a reference pole and as the sectorial origin. Since $z_0 = 0$ and $I_{yz} = 0$, the coordinates of A are given, according to Eqs. (2.12) and (2.13), by

$$y_A = y_O + \frac{I_{z\omega_O}}{I_y}$$
$$z_A = -\frac{I_{y\omega_O}}{I_z}$$

The sectorial product of area $I_{y\omega_O}$ is zero by symmetry, and the sectorial product of area $I_{z\omega_O}$ is

$$I_{z\omega_O} = \int z(s)\omega_O(s)dA = \int_0^b (-\frac{h}{2})(-\frac{h}{2}s_2)t_f\,ds_2 + \int_0^b (\frac{h}{2})(\frac{h}{2}s_4)t_f\,ds_4 = \frac{b^2h^2t_f}{4}$$

The area moment of inertia I_y for this section is

$$I_y = \frac{6bh^2t_f + h^3t_w}{12}$$

The coordinates of the principal pole A are found to be

$$y_A = y_O + \frac{3b^2t_f}{6bt_f + ht_w}$$
$$z_A = 0$$

The principal origin s_0 for the principal pole A of the symmetric channel section is determined from the condition

$$Q_{\omega_A} - \omega_A(s_0)A = 0$$

where ω_A has the arbitrarily selected origin O and is given by

$$\omega_A(s_1) = (y_A - y_O)s_1$$
$$\omega_A(s_2) = (y_A - y_O)\frac{h}{2} - \frac{h}{2}s_2$$
$$\omega_A(s_3) = -(y_A - y_O)s_3$$
$$\omega_A(s_4) = -(y_A - y_O)\frac{h}{2} + \frac{h}{2}s_4$$

The first moment of sectorial area with pole A and origin at O happens to be zero by symmetry

$$Q_{\omega_A} = \int \omega_A(s)dA = 0$$

so that the point O is the principal origin for the principal pole A, and ω_A is the principal warping function.

The *warping constant* I_ω is defined as the sectorial moment of inertia of the principal warping function

$$I_\omega = \int \omega^2(s)dA$$

Because the section shown in Figure 2.6 is symmetric with respect to the y axis, the warping constant is calculated as follows

$$I_\omega = 2\int_0^{h/2} \omega_A(s_1)^2 t_w \, ds_1 + 2\int_0^{b} \omega_A(s_2)^2 t_f \, ds_2 = \frac{b^3 h^2 t_f (3bt_f + 2ht_w)}{12(6bt_f + ht_w)}$$

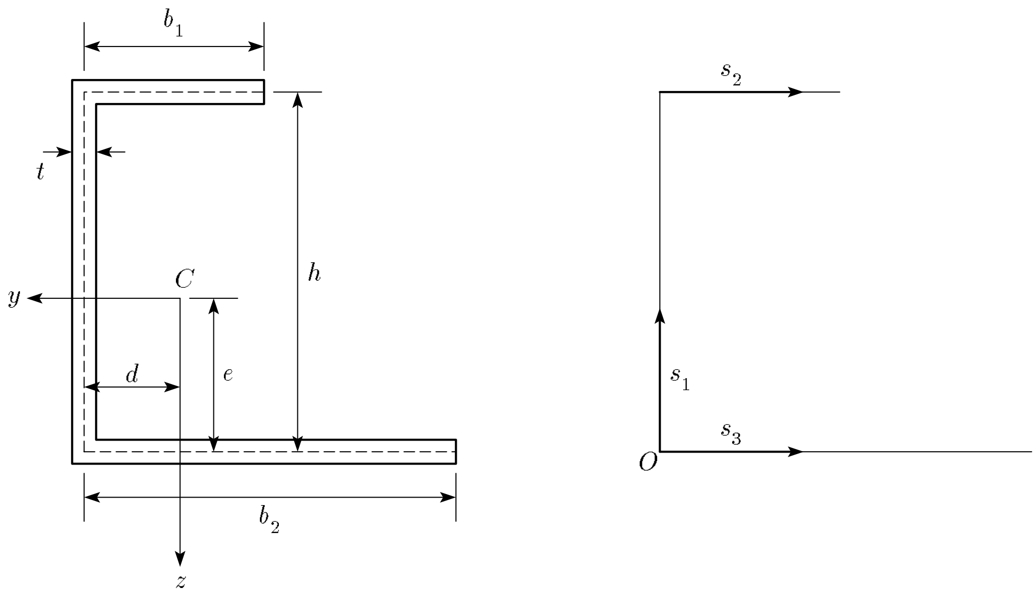

FIGURE 2.7 *Unsymmetric channel section*

For the unsymmetric channel section shown in Figure 2.7 with the dimensions

$$b_1 = b \qquad b_2 = 2b \qquad h = 2b$$

and constant thickness t, the centroid C is at a horizontal distance d and a vertical distance e from the intersection O of the lower flange and the web

$$d = \frac{b}{2} \qquad e = \frac{4b}{5}$$

The area moments of inertia and the area product of inertia are

$$I_y = \frac{52tb^3}{15} \qquad I_z = \frac{7tb^3}{4} \qquad I_{yz} = -tb^3$$

If the point O is used both as the pole and the sectorial origin, the warping function is

$$\omega_O(s_1) = 0 \qquad \omega_O(s_2) = -hs \qquad \omega_O(s_3) = 0$$

To find the principal pole, the values of the sectorial products of area are needed

$$I_{y\omega_O} = \int y\omega_O dA = \int_0^b (d - s_2)\omega_O(s_2)tds_2 = \int_0^b tbs_2(2s_2 - b)ds_2 = \frac{tb^4}{6}$$

$$I_{z\omega_O} = \int z\omega_O dA = \int_0^b -(h - e)\omega_O(s_2)tds_2 = \int_0^b \frac{12tb^2 s_2}{5}ds_2 = \frac{6tb^4}{5}$$

The principal pole A has the coordinates

$$y_A = y_O + \frac{I_{z\omega_O}I_z - I_{y\omega_O}I_{yz}}{I_y I_z - I_{yz}^2} = \frac{18b}{19}$$

$$z_A = z_O + \frac{I_{z\omega_O}I_{yz} - I_{y\omega_O}I_y}{I_y I_z - I_{yz}^2} = \frac{128b}{285}$$

The warping function with principal pole A and origin O is

$$\omega_A(s_1) = (y_A - y_O)s_1 = \frac{17b}{38}s_1$$

$$\omega_A(s_2) = (y_A - y_O)h - (h - e + z_A)s_2 = \frac{17b^2}{19} - \frac{94b}{57}s_2$$

$$\omega_A(s_3) = (e - z_A)s_3 = \frac{20b}{57}s_3$$

The first sectorial area moment is

$$Q_{\omega_A} = \int_0^h \omega_A(s_1)tds_1 + \int_0^{b_1} \omega_A(s_2)tds_2 + \int_0^{b_2} \omega_A(s_3)tds_3 = \frac{5tb^3}{3}$$

The condition for s_0 to be a principal origin is

$$Q_{\omega_A} - \omega_A(s_0)A = \frac{5tb^3}{3} - 5tb\omega_A(s_0) = 0$$

or

$$\omega_A(s_0) = \frac{b^2}{3}$$

According to Eq. (2.8), the shift of the origin to s_0 gives the principal warping function

$$\omega(s) = \omega_A(s) - \omega_A(s_0) = \omega_A(s) - \frac{b^2}{3}$$

which, when written out for the web and the flanges, yields

$$\omega(s_1) = \omega_A(s_1) - \frac{b^2}{3} = \frac{17b}{38}s_1 - \frac{b^2}{3}$$

$$\omega(s_2) = \omega_A(s_2) - \frac{b^2}{3} = -\frac{94b}{57}s_2 - \frac{32b^2}{57}$$

$$\omega(s_3) = \omega_A(s_3) - \frac{b^2}{3} = \frac{20b}{57}s_3 - \frac{b^2}{3}$$

The principal warping function $\omega(s)$ shown sketched to scale in Figure 2.8. The function is zero at three distinct points of the median line of the section. Any one of these points can be regarded as the principal sectorial origin s_0.

If a cross section has a symmetry axis, say the y axis, this is a principal axis, so that $I_{yz} = 0$. In calculating the coordinates of the principal pole A, using Eqs. (2.12) and (2.13), the choice of the reference pole B is arbitrary. When B is chosen to

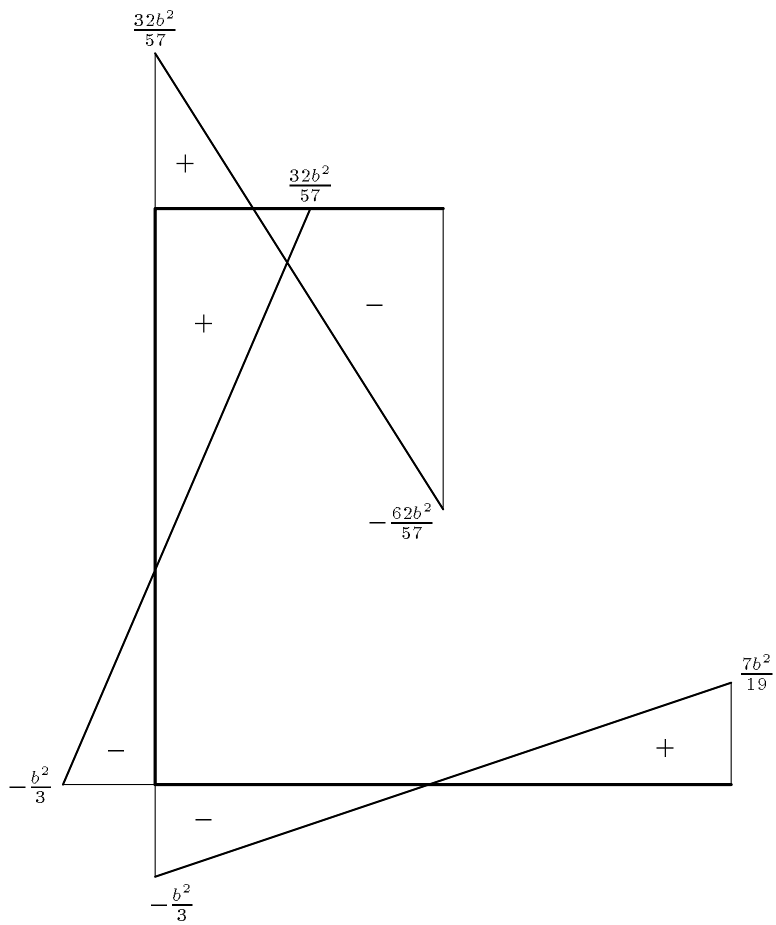

FIGURE 2.8 *Warping function for the unsymmetric channel section*

be any point on the symmetry axis y, it is readily verified that $I_{y\omega_B} = 0$. The coordinate z_A is then

$$z_A = z_B + \frac{I_{z\omega_B} I_{yz} - I_{y\omega_B} I_y}{I_y I_z - I_{yz}^2} = 0$$

This shows that the principal pole lies on the symmetry axis. In addition, the point of intersection of the y axis with the median line is a principal sectorial origin. For a doubly symmetric cross section, the point of intersection of the two axes of symmetry is the principal pole, the principal origin, and the centroid.

A formula for the warping constant in terms of the warping function ω_B, whose pole and origin are arbitrary, can be derived starting from the transformation equation Eq. (2.7)

$$\omega_A = \omega_B - \omega_B(s_0) + (z_A - z_B)(y - y_0) - (y_A - y_B)(z - z_0) \tag{2.15}$$

Since the origin s_0 for ω_A is a principal origin, the area integral of Eq. (2.15) is

$$0 = Q_{\omega_A} = Q_{\omega_B} - \omega_B(s_0)A - y_0(z_A - z_B)A + z_0(y_A - y_B)A \tag{2.16}$$

The first moments of area Q_y and Q_z have been set equal to zero in the calculation of the right side of this equation because y and z are centroidal axes. The result of Eq. (2.16) allows Eq. (2.15) to be rewritten as

$$\omega_A = \omega_B - \frac{Q_{\omega_B}}{A} + (z_A - z_B)y - (y_A - y_B)z \tag{2.17}$$

The conditions for A to be a principal pole are then

$$I_{y\omega_A} = I_{y\omega_B} + (z_A - z_B)I_z - (y_A - y_B)I_{yz} = 0 \tag{2.18}$$

$$I_{z\omega_A} = I_{z\omega_B} + (z_A - z_B)I_{yz} - (y_A - y_B)I_y = 0 \tag{2.19}$$

The warping constant is given by

$$\begin{aligned} I_\omega = {} & I_{\omega_B} + 2(z_A - z_B)I_{y\omega_B} - 2(y_A - y_B)I_{z\omega_B} - \frac{Q_{\omega_B}^2}{A} \\ & + (y_A - y_B)^2 I_y + (z_A - z_B)^2 I_z - 2(y_A - y_B)(z_A - z_B)I_{yz} \end{aligned}$$

which is simplified by using Eqs. (2.18) and (2.19) to

$$I_\omega = I_{\omega_B} - \frac{Q_{\omega_B}^2}{A} - (y_A - y_B)^2 I_y + 2(y_A - y_B)(z_A - z_B)I_{yz} - (z_A - z_B)^2 I_z \tag{2.20}$$

The channel section with double flanges shown in Figure 2.9 has uniform thickness t, and it is symmetric with respect to the y axis. The point O, which is the point of intersection of the median line and the y axis, is chosen as a convenient pole and origin for the warping function

$$\begin{aligned} \omega_O(s_1) &= 0 & 0 \leqslant s_1 \leqslant \frac{h_2}{2} \\ \omega_O(s_2) &= -\frac{h_1}{2}s_2 & 0 \leqslant s_2 \leqslant b \\ \omega_O(s_3) &= -\frac{h_2}{2}s_3 & 0 \leqslant s_3 \leqslant b \end{aligned}$$

The area moment of inertia I_y is calculated using the centerline dimensions

$$I_y = \frac{th_2^3}{12} + \frac{tbh_1^2}{2} + \frac{tbh_2^2}{2}$$

and the sectorial product of area $I_{z\omega_O}$ is found, taking advantage of symmetry,

$$\begin{aligned} I_{z\omega_O} &= 2\int_0^b -\frac{h_1}{2}\omega_O(s_1)tds_1 + 2\int_0^b -\frac{h_2}{2}\omega_O(s_1)tds_1 \\ &= \frac{tb^2}{4}(h_1^2 + h_2^2) \end{aligned}$$

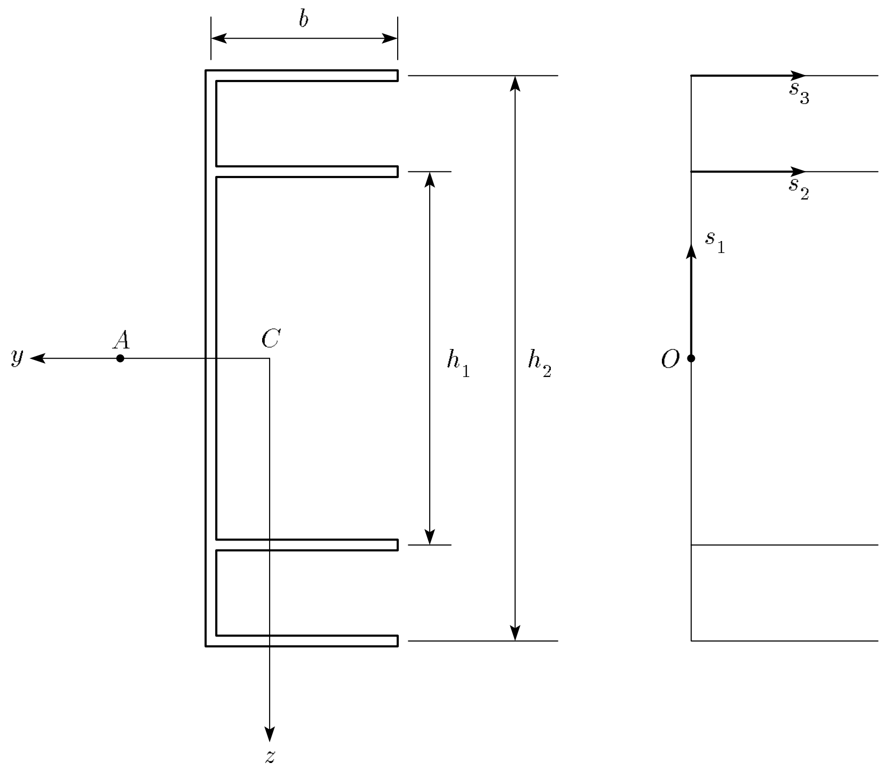

FIGURE 2.9 *Channel section with double flanges*

The principal pole A is on the symmetry axis y

$$y_A = y_O + \frac{I_{z\omega_O}}{I_y} = y_O + \frac{3b^2(h_1^2+h_2^2)}{h_2^2+6b(h_1^2+h_2^2)}$$

The warping constant is given by Eq. (2.20)

$$I_\omega = I_{\omega_O} - (y_A - y_O)^2 I_y = \frac{tb^3(h_1^2+h_2^2)(2h_2^3+3b(h_1^2+h_2^2))}{12(h_2^3+6b(h_1^2+h_2^2))}$$

Since point O is the principal origin, the principal warping function can be written as

$$\omega(s_1) = (y_A - y_O)s_1$$
$$\omega(s_2) = (y_A - y_O)\frac{h_1}{2} - \frac{h_1}{2}s_2$$
$$\omega(s_3) = (y_A - y_O)\frac{h_2}{2} - \frac{h_2}{2}s_3$$

from which the value found for I_ω from Eq. (2.20) can be verified by evaluating

$$I_\omega = 2\int_0^{h_2/2}\omega(s_1)^2 t ds_1 + 2\int_0^b \omega(s_2)^2 t ds_2 + 2\int_0^b \omega(s_3)^2 t ds_3$$

2.3. Stress-Strain Relations

The only significant stresses will be assumed to be the normal stress σ_x and the shear stress τ_{xs}. Although the strain γ_{xs} was taken to be negligible in Section 2.1, the corresponding stress τ_{xs} is not assumed to be zero, and it will be noticed that this contradicts Hooke's law. The distribution of normal stress across the wall thickness will be assumed to be uniform. The shear stress due to unrestrained, or Saint-Venant, torsion will be assumed to be linear, with a zero value at the median line. All other shear stresses will be assumed to be constant across the wall thickness.

According to the kinematic assumption that the median line of the cross section is inextensible, the strain ϵ_s is zero

$$\epsilon_s = \frac{1}{E}(\sigma_s - \nu\sigma_x) = 0$$

where E is the modulus of elasticity and ν is Poisson's ratio. The longitudinal strain is

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_s) = \frac{1-\nu^2}{E}\sigma_x$$

The normal stress σ_x is written, using the kinematical expression for ϵ_x given in Eq. (2.6), as

$$\sigma_x = \bar{E}\big(u_0'(x) - v_A''(x)y(s) - w_A''(x)z(s) - \theta_x''(x)\omega_A(s)\big) \tag{2.21}$$

in which the material constant $\bar{E}$ is

$$\bar{E} = \frac{E}{1-\nu^2}$$

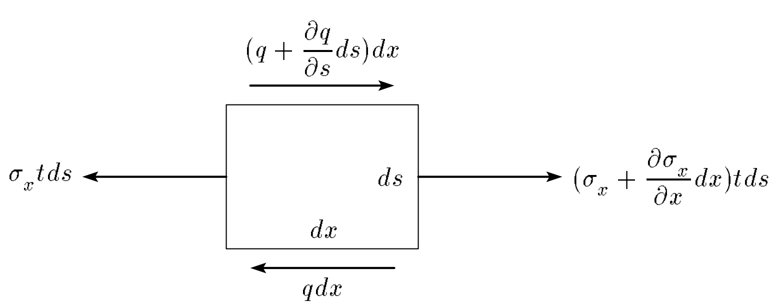

FIGURE 2.10 *Forces on a wall element*

The shear stress is determined from the x component of the force equilibrium equation for the wall element shown in Figure 2.10. If it is assumed that there is no longitudinal applied load on the surface

$$\frac{\partial q}{\partial s} + t(s)\frac{\partial \sigma_x}{\partial x} = 0$$

where t is the thickness of the wall, and the shear flow q is defined by

$$q(x,\, s) = \tau_{xs}(x,\, s)t(s)$$

The shear flow is found by integrating the equilibrium equation with respect to s

$$q(x,\, s) = q_0(x) - \int_0^s \frac{\partial \sigma_x}{\partial x} t(s) ds \tag{2.22}$$

where $q_0(x)$ is the shear flow at $s = 0$. By substituting the expression

$$\frac{\partial \sigma_x}{\partial x} = \bar{E}\big(u_0''(x) - v_A'''(x)y(s) - w_A'''(x)z(s) - \theta_x'''(x)\omega_A(s)\big) \tag{2.23}$$

into Eq. (2.22) and writing $dA = t(s)ds$ for the element of cross-sectional area, the shear flow can be written as

$$q(x,\, s) = q_0(x) + \bar{E}\big(v_A'''(x)Q_z(s) + w_A'''(x)Q_y(s) + \theta_x'''(x)Q_{\omega_A}(s) - u_0''(x)A(s)\big) \tag{2.24}$$

where

$$A(s) = \int_0^s t(s)ds = \int_0^s dA$$
$$Q_y(s) = \int_0^s z(s)dA$$
$$Q_z(s) = \int_0^s y(s)dA$$
$$Q_{\omega_A}(s) = \int_0^s \omega_A(s)dA$$

2.4. Equations of Equilibrium

The equations of equilibrium will be written for a differential element of length dx of the beam. It will be assumed that the coordinate axes y, z are centroidal, and that only the principal the warping function ω is being used. Let p_x be applied force per unit length of the beam in the longitudinal direction. The normal stress resultant on the differential element is

$$\int \big(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\big)dA - \int \sigma_x dA = dx \int \frac{\partial \sigma_x}{\partial x} dA$$

and the equilibrium of forces in the x direction gives

$$dx \int \frac{\partial \sigma_x}{\partial x} dA + p_x dx = 0$$

The first term is evaluated by integrating the expression on the right side of Eq. (2.23) over the cross sectional area. The result is

$$\bar{E}\big(u_0''(x)A - v_A'''(x)Q_z - w_A'''(x)Q_y + \theta_x'''(x)Q_\omega\big) + p_x = 0$$

which simplifies to

$$\bar{E}u_0''(x)A + p_x = 0$$

Let p_y denote the applied force in the y direction per unit length of the beam. The direct shear force in the y direction balances the applied force in this direction

$$\int \frac{\partial q}{\partial x} \cos\beta ds dx + p_y dx = 0$$

where β is the angle between the y and s axes, and q is the direct shear flow. Since $dy = ds\cos\beta$,

$$\int \frac{\partial q}{\partial x}\cos\beta ds = \frac{\partial q}{\partial x} y\bigg|_{\text{edges}} - \int y \frac{\partial}{\partial s}\frac{\partial q}{\partial x} ds = \int y \frac{\partial}{\partial x}\left(t\frac{\partial \sigma_x}{\partial x}\right) ds = \int y \frac{\partial^2 \sigma_x}{\partial x^2} dA$$

where it has been assumed that the edges are free of shear stresses. The last term is evaluated by differentiating the expression on the right side of Eq. (2.23) with respect to x and integrating the result over the cross sectional area

$$\begin{aligned}\int y \frac{\partial^2 \sigma_x}{\partial x^2} dA &= \bar{E}\big(u_0'''(x)Q_z - v_A^{iv}(x)I_z - w_A^{iv}(x)I_{yz} + \theta_x^{iv}(x)I_{y\omega}\big) \\ &= \bar{E}\big(-v_A^{iv}(x)I_z - w_A^{iv}(x)I_{yz}\big)\end{aligned}$$

Hence, the equation of equilibrium in the y direction becomes

$$\bar{E}I_z v_A^{iv}(x) + \bar{E}I_{yz} w_A^{iv}(x) = p_y \tag{2.25}$$

Let p_z denote the applied force in the z direction per unit length of the beam. The direct shear force in the z direction balances the applied force in this direction

$$\int \frac{\partial q}{\partial x} \sin\beta ds dx + p_z dx = 0$$

where β is the angle between the y and s axes, and q is the direct shear flow. Since $dz = ds\sin\beta$,

$$\int \frac{\partial q}{\partial x}\sin\beta ds = \frac{\partial q}{\partial x} z\bigg|_{\text{edges}} - \int z \frac{\partial}{\partial s}\frac{\partial q}{\partial x} ds = \int z \frac{\partial}{\partial x}\big(t\frac{\partial \sigma_x}{\partial x}\big) ds = \int z \frac{\partial^2 \sigma_x}{\partial x^2} dA$$

where it has been assumed that the edges are free of shear stresses. The last term is evaluated using Eq. (2.23)

$$\begin{aligned}\int z \frac{\partial^2 \sigma_x}{\partial x^2} dA &= \bar{E}\big(u_0'''(x)Q_y - v_A^{iv}(x)I_{yz} - w_A^{iv}(x)I_y + \theta_x^{iv}(x)I_{z\omega}\big) \\ &= \bar{E}\big(-v_A^{iv}(x)I_{yz} - w_A^{iv}(x)I_y\big)\end{aligned}$$

The equation of equilibrium in the z direction is

$$\bar{E}I_{yz} v_A^{iv}(x) + \bar{E}I_y w_A^{iv}(x) = p_z \tag{2.26}$$

The total torque at the section is the sum of two parts

$$T = T_t + T_\omega$$

The torque T_t is due to the shear stresses resulting from pure, or unrestrained, torsion. It is related to the angle θ_x of rotation by

$$T_t = GJ\theta_x'(x)$$

The torque T_ω is called the warping torque. It is due to the shear flow q. For a beam element of length dx, equilibrium of the torques about the pole A gives

$$\mathbf{e}_x \cdot \int \mathbf{r}_A \times \frac{\partial q}{\partial x} dx ds \mathbf{e}_s + GJ\theta_x'' dx + m(x)dx = 0$$

where $\mathbf{r}_A$ is the vector from the pole A to the point at s, and $m(x)$ is the applied torsional moment per unit length. Since

$$(\mathbf{r}_A \times \mathbf{e}_s) \cdot \mathbf{e}_x = \mathbf{r}_A \cdot (\mathbf{e}_s \times \mathbf{e}_x) = -\mathbf{r}_A \cdot \mathbf{e}_n$$

the first term in the torque equation can be rewritten as

$$\mathbf{e}_x \cdot \int \mathbf{r}_A \times \frac{\partial q}{\partial x} dx ds \mathbf{e}_s = -\int \frac{\partial q}{\partial x} dx \mathbf{r}_A \cdot \mathbf{e}_n ds = -\int \frac{\partial q}{\partial x} dx d\omega$$

An integration by parts gives

$$\int \frac{\partial q}{\partial x} d\omega = \omega \frac{\partial q}{\partial x}\bigg|_{\text{edges}} - \int \omega \frac{\partial}{\partial s}\frac{\partial q}{\partial x} ds = \int \omega \frac{\partial^2 \sigma_x}{\partial x^2} t(s) ds$$

so that the torsional equilibrium equation becomes

$$-\int \omega \frac{\partial^2 \sigma_x}{\partial x^2} dA + GJ\theta_x'' + m(x) = 0$$

which is brought to its final form using Eq. (2.23)

$$\bar{E} I_\omega \theta_x^{iv} - GJ\theta_x'' = m(x) \tag{2.27}$$

2.5. Stress Resultants

The stress resultants for the normal stress σ_x are the axial force N, the bending moments M_y and M_z, and the *bimoment* M_ω, which are defined by

$$N = \int \sigma_x dA$$

$$M_y = \int z\sigma_x dA$$

$$M_z = -\int y\sigma_x dA$$

$$M_\omega = \int \omega\sigma_x dA$$

The normal stress is given by Eq. (2.21)

$$\sigma_x = \bar{E}\left(u_0'(x) - v_A''(x)y(s) - w_A''(x)z(s) - \theta_x''(x)\omega(s)\right)$$

The stress resultants are evaluated, recalling that the origin of the coordinates y, z is the centroid, and that ω is the principal warping function

$$\begin{pmatrix} N \\ M_y \\ M_z \\ M_\omega \end{pmatrix} = \bar{E} \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -I_{yz} & -I_y & 0 \\ 0 & I_z & I_{yz} & 0 \\ 0 & 0 & 0 & -I_\omega \end{pmatrix} \begin{pmatrix} u_0' \\ v_A'' \\ w_A'' \\ \theta_x'' \end{pmatrix}$$

Hence

$$\bar{E}u_0'(x) = \frac{N(x)}{A}$$
$$\bar{E}v_A''(x) = \frac{I_{yz}M_y(x) + I_yM_z(x)}{I_yI_z - I_{yz}^2}$$
$$\bar{E}w_A''(x) = -\frac{I_zM_y(x) + I_{yz}M_z(x)}{I_yI_z - I_{yz}^2}$$
$$\bar{E}\theta_x''(x) = -\frac{M_\omega(x)}{I_\omega}$$

The normal stress is found in terms of the stress resultants by using these expressions in Eq. (2.21)

$$\sigma_x = \frac{N}{A} - \frac{I_{yz}M_y + I_yM_z}{I_yI_z - I_{yz}^2}y + \frac{I_zM_y + I_{yz}M_z}{I_yI_z - I_{yz}^2}z + \frac{M_\omega \omega}{I_\omega} \tag{2.28}$$

In Eq. (2.24), let the point $s = 0$ be placed at the free edge so that the shear flow $q_0(x)$ is zero, and suppose that there is no longitudinal external load p_x on the beam. Then Eq. (2.4) shows that $u_0''(x)$ is zero, and the shear flow is given by

$$q(x,\, s) = \bar{E}\big(v_A'''(x)Q_z(s) + w_A'''(x)Q_y(s) + \theta_x'''(x)Q_\omega(s)\big) \tag{2.29}$$

Let the shear stress resultants V_y, V_z, and T_ω be defined by

$$\begin{aligned} V_y &= \int q(x,\, s)ds\cos\beta(s) = \int q(x,\, s)dy \\ V_z &= \int q(x,\, s)ds\sin\beta(s) = \int q(x,\, s)dz \\ T_\omega &= \int q(x,\, s)d\omega \end{aligned} \tag{2.30}$$

The definition of $Q_z(s)$ is

$$Q_z(s) = \int_0^s y(s)dA$$

Integration by parts gives

$$\int Q_z dz = zQ_z\Big|_A - \int zdQ_z = -\int zydA = -I_{yz}$$

Similarly,

$$\int Q_z dy = yQ_z\Big|_A - \int ydQ_z = -\int y^2 dA = -I_z$$

and

$$\int Q_z d\omega = \omega Q_z\Big|_A - \int \omega ydA = -I_{y\omega} = 0$$

The corresponding integrals for $Q_y(s)$ are

$$\int Q_y dy = -I_{yz}$$
$$\int Q_y dz = -I_y$$
$$\int Q_y d\omega = 0$$

The definition of $Q_\omega(s)$ is

$$Q_\omega(s) = \int_0^s \omega(s) dA$$

Integration by parts gives

$$\int Q_\omega dz = zQ_\omega \Big|_A - \int z dQ_\omega = -\int z\omega dA = -I_{z\omega} = 0$$

Similarly,

$$\int Q_\omega dy = -I_{y\omega} = 0$$

and

$$\int Q_\omega d\omega = \omega Q_\omega \Big|_A - \int \omega^2 dA = -I_\omega$$

The stress resultants are evaluated using Eq. (2.30) and Eq. (2.29)

$$\begin{aligned} V_y &= -\bar{E}\left(I_z v_A'''(x) + I_{yz} w_A'''(x)\right) \\ V_z &= -\bar{E}\left(I_{yz} v_A'''(x) + I_y w_A'''(x)\right) \\ T_\omega &= -\bar{E} I_\omega \theta_x'''(x) \end{aligned}$$

Substitution of these results into Eq. (2.29) gives the shear flow

$$q(x,\, s) = -\frac{I_y Q_z(s) - I_{yz} Q_y(s)}{I_y I_z - I_{yz}^2} V_y - \frac{I_z Q_y(s) - I_{yz} Q_z(s)}{I_y I_z - I_{yz}^2} V_z - \frac{Q_\omega(s)}{I_\omega} T_\omega \qquad (2.31)$$

The total shear stress is found by adding Saint-Venant's torsional stress to the contribution from $q(x,\, s)$

$$\tau_{xs} = \frac{2T_t n}{J} + \frac{q}{t} \qquad (2.32)$$

where n is the coordinate measured from the median line in the normal direction.

2.6. Shear Center

A beam is said to be in *pure flexure* if the angle of twist $\theta_x(x)$ is identically zero. The *shear center* S is defined as the point, in the plane of the cross section, through which the line of action of the transverse shear forces must pass for the beam to be in pure flexure. Suppose that a beam is in pure flexure under the action of transverse shear forces in the y direction only, as shown in Figure 2.11. The line of action of V_y passes through the shear center S. From Figure 2.11, the moment

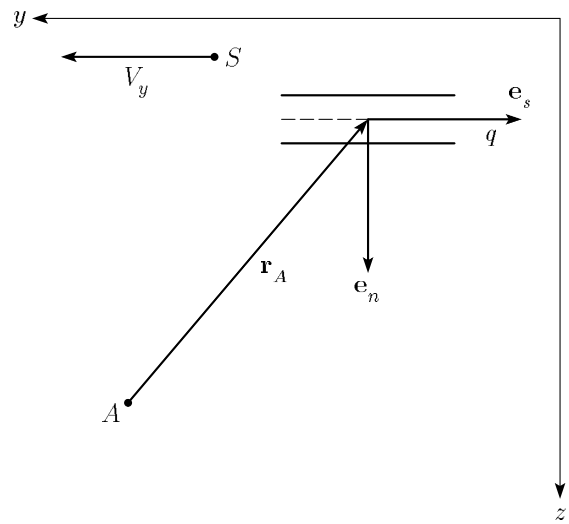

FIGURE 2.11 *Shear center calculation*

of the shear stresses about point A, which is the principal pole of the warping function ω, is calculated by integrating

$$dM_A = \mathbf{e}_x \cdot (\mathbf{r}_A \times q ds \mathbf{e}_s) = \mathbf{r}_A \cdot (\mathbf{e}_s \times \mathbf{e}_x) q ds = \mathbf{r}_A \cdot \mathbf{e}_n q ds = q d\omega$$

over the cross-sectional area

$$M_A = \int q(x, s) d\omega = T_\omega = -\bar{E} I_\omega \theta_x'''(x) = 0$$

This moment is equal to the moment of V_y about A

$$M_A = \mathbf{e}_x \cdot (\mathbf{r}_{SA} \times V_y \mathbf{e}_y) = (z_A - z_S) V_y = 0$$

which shows that the z coordinates of the shear center and the principal pole are identical. A similar computation with the beam cross section subjected only to V_z shows that the y coordinates of A and S are also identical. The shear center and the principal pole are, therefore, the same point.

2.7. Calculation of the Angle of Twist

The warping stresses in a thin-walled beam depend on the bimoment M_ω and the warping torsion T_ω

$$M_\omega(x) = -\bar{E} I_\omega \theta_x''(x)$$
$$T_\omega(x) = -\bar{E} I_\omega \theta_x'''(x)$$

The torque equilibrium equation 2.27, solved with the applicable boundary conditions, determines the angle of twist as as function of x. With the definition

$$c^2 = \frac{GJ}{\bar{E} I_\omega}$$

Eq. (2.27) is rewritten in the form

$$\frac{d^4\theta_x}{dx^4} - c^2\frac{d^2\theta_x}{dx^2} = \frac{m(x)}{\bar{E}I_\omega} \tag{2.33}$$

The most common boundary conditions on the angle of twist are those for *fixed, simple, free* or beam supports. At a fixed support, no twisting or warping occurs. These kinematical conditions are expressed by

$$\theta_x = 0 \qquad \theta'_x = 0$$

where the second condition is obtained by setting equal to zero the warping component, which is proportional to θ'_x, of the longitudinal displacement $u(x, s)$. A simple support does not allow twisting and is free of normal stress

$$\theta_x = 0 \qquad \theta''_x = 0$$

where the second condition expresses that the bimoment is zero

$$M_\omega = \int \omega\sigma_x dA = 0$$

At a free support there are two statical conditions, one expressing that there is no normal stress, and the other that the total torque is zero. The second of these conditions is

$$T_t + T_\omega = GJ\theta'_x - \bar{E}I_\omega\theta'''_x = \bar{E}I_\omega(c^2\theta'_x - \theta'''_x) = 0$$

Thus, for a free support, the boundary conditions are

$$\theta''_x = 0 \qquad c^2\theta'_x - \theta'''_x = 0$$

The general solution of Eq. (2.33) is

$$\theta_x(x) = C_1 + C_2 x + C_3\cosh cx + C_4\sinh cx - \frac{1}{cGJ}\int_0^x [c(x-\xi) - \sinh c(x-\xi)]m(\xi)d\xi \tag{2.34}$$

where C_k, $1 \leqslant k \leqslant 4$, are the constants of integration, and one end of the beam is assumed to be at $x = 0$. The bimoment is obtained from Eq. (2.34) by two differentiations

$$M_\omega(x) = -JG(C_3\cosh cx + C_4\sinh cx) - \frac{1}{c}\int_0^x m(\xi)\sinh c(x-\xi)d\xi \tag{2.35}$$

The warping torque is the derivative of $M_\omega(x)$

$$T_\omega(x) = -cGJ(C_3\sinh cx + C_4\cosh cx) - \int_0^x m(\xi)\cosh c(x-\xi)d\xi \tag{2.36}$$

The pure torsion torque is

$$T_t(x) = GJ(C_2 + cC_3\sinh cx + cC_4\cosh cx) - \int_0^x m(\xi)[1 - \cosh c(x-\xi)]d\xi \tag{2.37}$$

and the total torque is given by

$$T(x) = GJC_2 - \int_0^x m(\xi)d\xi \tag{2.38}$$

A concentrated torque is applied at the unsupported end of the cantilever beam shown in Figure 2.12. For this loading condition, the distributed torque

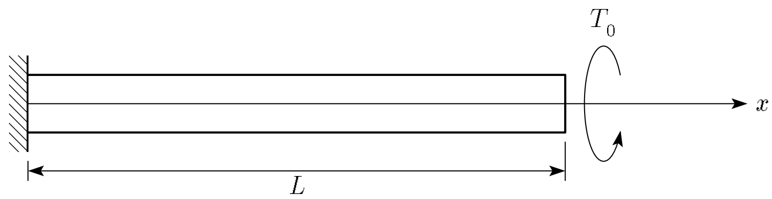

FIGURE 2.12 *Cantilever beam with end torque*

$m(x)$ is zero, because there is no external torque for the cross sections that lie between the two end sections. The external torque is set equal to the total torque at $x = L$

$$T(L) = GJC_2 = T_0$$

The other boundary condition at $x = L$ is that the cross section is free of normal stress

$$M_\omega(L) = -GJ(C_3 \cosh cL + C_4 \sinh cL) = 0$$

At the fixed end the boundary conditions are

$$\theta_x(0) = C_1 + C_3 = 0$$
$$\theta'_x(0) = C_2 + cC_4 = 0$$

The angle of twist is determined from these boundary conditions

$$\theta_x(x) = \frac{T_0}{cGJ}\big(cx - \sinh cx - \tanh cL(1 - \cosh cx)\big)$$

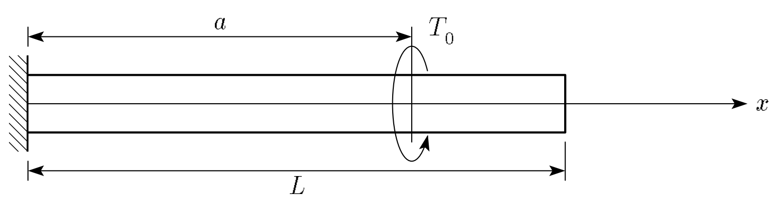

FIGURE 2.13 *Cantilever beam with torque at* $x = a$

If the torque is applied at a point $x = a < L$ as shown in Figure 2.13, the distributed torque is no longer zero. The concentrated torque of magnitude T_0 can be expressed as a distributed torque in terms of the Dirac delta function

$$m(x) = T_0\delta(x - a)$$

The angle of twist is calculated from Eq. (2.34) as

$$\theta_x(x) = C_1 + C_2 x + C_3 \cosh cx + C_4 \sinh cx - \frac{T_0}{cGJ}[c(x-a) - \sinh c(x-a)]U(x-a)$$

where U denotes the unit step function. The boundary conditions are

$$\theta_x(0) = C_1 + C_3 = 0$$
$$\theta'_x(0) = C_2 + cC_4 = 0$$
$$T(L) = GJC_2 - T_0 = 0$$
$$M_\omega(L) = -JG(C_3 \cosh cL + C_4 \sinh cL) - \frac{T_0}{c} \sinh c(L-a) = 0$$

The angle of twist for $0 \leqslant x \leqslant a$ is

$$\theta_x^L(x) = \frac{T_0}{cGJ}\left[cx - \sinh cx + (1-\cosh cx)\left(\frac{\sinh c(L-a)}{\cosh cL} - \tanh cL\right)\right]$$

and for $a \leqslant x \leqslant L$

$$\theta_x^R(x) = \theta_x^L(x) - \frac{T_0}{cGJ}[c(x-a) - \sinh c(x-a)]$$

2.8. Stress Analysis

As a first example, stresses in a cantilever beam of length L, with its fixed end at $x = 0$ and its free end at $x = L$, will be analyzed. The cross section of the beam, shown in Figure 2.14, is symmetric with respect to the y axis and is of constant thickness t. The load is a single vertical force of magnitude P applied at the free end of the beam. The point of application of P on the cross section is the lower end of the left flange.

The centroid is located by the dimension a

$$a = \frac{h(2b_2 + h)}{2(h + b_1 + b_2)}$$

The area moments of inertia are

$$I_y = \frac{t}{12}(b_1^3 + b_2^3)$$
$$I_z = tb_1 a^2 + tb_2(h-a)^2 + \frac{t}{3}(a^3 + (h-a)^3)$$
$$I_{yz} = 0$$

The warping function whose origin and pole are both chosen to be point O can be written from Figure 2.14 as

$$\omega_O(s_4) = -hs_4 \qquad \omega_O(s_5) = hs_5$$

The warping function ω_O is zero on the branches s_1, s_2, and s_3. To calculate the y coordinate of the shear center using Eq. (2.12), its is necessary to evaluate the sectorial product of area

$$I_{z\omega_O} = \int z(s)\omega_O(s)dA = 2\int_0^{b_1/2} hs_4^2 t ds_4 = \frac{thb_1^3}{12}$$

The shear center location is then given by Eq. (2.12) as

$$y_S = y_O + \frac{I_{z\omega_O}}{I_y} = y_O + \frac{hb_1^3}{b_1^3 + b_2^3}$$

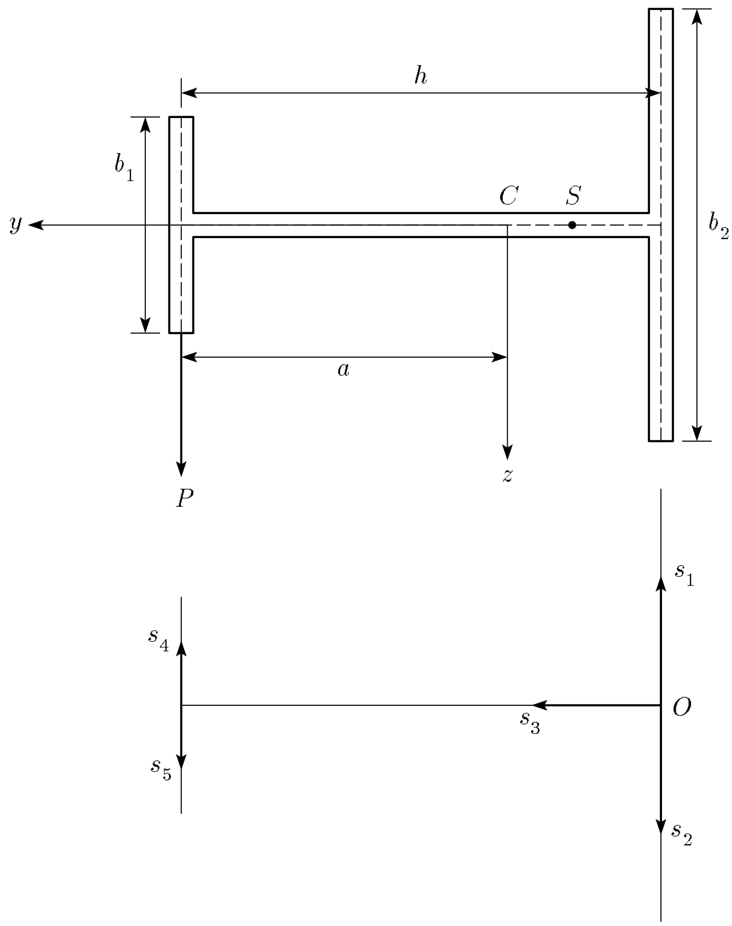

FIGURE 2.14 *I-beam cross section*

or

$$y_S = -(h-a) + \frac{hb_1^3}{b_1^3+b_2^3} = -\frac{h^2(b_2^2-b_1^2)+2hb_1b_2(b_2^2-b_1^2)}{2(b_1+b_2+h)(b_1^3+b_2^3)}$$

which shows that S is to the left of the centroid for $b_2 > b_1$. The warping constant I_ω is found from Eq. (2.20), which for this cross section becomes

$$I_\omega = I_{\omega_O} - (y_S - y_O)^2 I_y = 2\int_0^{b_1/2} h^2 s_4^2 t\, ds_4 - \frac{h^2 b_1^6}{(b_1^3+b_2^3)^2}\frac{t}{12}(b_1^3+b_2^3) = \frac{th^2}{12}\frac{b_1^3 b_2^3}{b_1^3+b_2^3}$$

The principal warping function, the origin of which can be taken at O, is found by transforming ω_O according to Eq. (2.9)

$$\omega(s) = \omega_O(s) - (y_S - y_O)z(s)$$

In terms of the branch coordinates s_k, $1 \leqslant k \leqslant 5$, defined in Figure 2.14, the principal warping function is

$$\begin{aligned}
\omega(s_1) &= -(y_S - y_O)z(s_1) = (y_S - y_O)s_1 \\
\omega(s_2) &= -(y_S - y_O)z(s_2) = -(y_S - y_O)s_2 \\
\omega(s_3) &= 0 \\
\omega(s_4) &= -hs_4 - (y_S - y_O)z(s_4) = -(h - y_S + y_O)s_4 \\
\omega(s_5) &= hs_5 - (y_S - y_O)z(s_5) = (h - y_S + y_O)s_5
\end{aligned}$$

The applied force P at the free end of the beam does not pass through the shear center. The force-couple equivalent of P at the shear center S is the force P and the torsional moment T_0 of P about S

$$T_0 = (a + |y_S|)P = \frac{Phb_2^3}{b_1^3 + b_2^3}$$

The angle of twist of the beam is, therefore, determined as for the beam of Figure 2.12

$$\theta_x(x) = \frac{T_0}{cGJ}\big(cx - \sinh cx - \tanh cL(1 - \cosh cx)\big)$$

For the torsional constant J, Saint-Venant's approximation can be used

$$J = \frac{t^3}{3}(h + b_1 + b_2)$$

The constant c depends on material constants and cross-sectional dimensions

$$c^2 = \frac{GJ}{\bar{E}I_\omega} = \frac{Gt^3(h + b_1 + b_2)}{3\bar{E}h}(1 + \frac{b_2^3}{b_1^3})$$

The internal forces at the clamped end are

$$V_z = P \qquad M_y = -PL \qquad T = T_0$$

The torsional shear stress, which is proportional to θ_x', is zero at the clamped end. The shear stress distribution over the cross section at the fixed end $x = 0$ is given by Eq. (2.32) as

$$\tau_{xs}(s) = -\frac{PQ_y(s)}{tI_y} - \frac{T_\omega Q_\omega(s)}{tI_\omega} \tag{2.39}$$

In Eq. (2.39), the first moment area $Q_y(s)$ is calculated using section cuts such as those indicated in Figure 2.15. For instance, with the section cut on the right flange

$$Q_y(z) = \int_{-b_2/2}^{z} zt\,dz = \frac{t}{2}(z^2 - \frac{b_2^2}{4})$$

The corresponding shear stress distribution is

$$\tau_1(z) = \frac{P}{2I_y}(\frac{b_2^2}{4} - z^2)$$

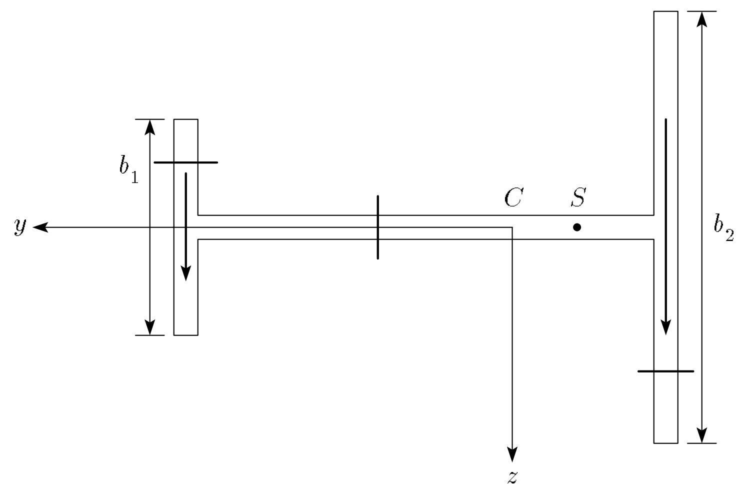

FIGURE 2.15 *Transverse shear flow directions and section cuts*

Similarly, the shear stress in the left flange is given by

$$\tau_1(z) = \frac{P}{2I_y}(\frac{b_1^2}{4} - z^2)$$

and the shear stress in the web is zero.

The second contribution to the shear stress in Eq. (2.39) is the *warping shear stress*

$$\tau_2(s) = -\frac{T_\omega Q_\omega(s)}{tI_\omega}$$

where T_ω is equal to the entire torque T_0, since the pure torsion torque T_t is zero at the clamped end. The first sectorial area moment $Q_\omega(s)$ of the principal warping function is calculated for the part of the cross section cut off at s, remembering that the integration starts at a free edge. The section cuts indicated in Figure 2.15 may be used for this calculation. For instance, with the section cut on the right flange

$$Q_\omega(s_2) = \int_{b_2/2}^{s_2} -(y_S - y_O)s_2 t ds_2 = \frac{t(y_S - y_O)}{2}(\frac{b_2^2}{4} - s_2^2)$$

Similarly

$$Q_\omega(s_1) = \int_{b_2/2}^{s_1} (y_S - y_O)s_1 t ds_1 = -\frac{t(y_S - y_O)}{2}(\frac{b_2^2}{4} - s_1^2)$$
$$Q_\omega(s_3) = 0$$
$$Q_\omega(s_4) = \int_{b_1/2}^{s_4} (y_S - y_O - h)s_4 t ds_4 = \frac{t(h - y_S + y_O)}{2}(\frac{b_1^2}{4} - s_4^2)$$
$$Q_\omega(s_5) = \int_{b_1/2}^{s_5} (h + y_O - y_S)s_5 t ds_5 = -\frac{t(h - y_S + y_O)}{2}(\frac{b_1^2}{4} - s_5^2)$$

The shear stresses are expressed by

$$\tau_2(s_1) = \frac{6T_0}{thb_2^3}(\frac{b_2^2}{4} - s_1^2)$$
$$\tau_2(s_2) = -\frac{6T_0}{thb_2^3}(\frac{b_2^2}{4} - s_2^2)$$
$$\tau_2(s_3) = 0$$
$$\tau_2(s_4) = -\frac{6T_0}{thb_1^3}(\frac{b_1^2}{4} - s_4^2)$$
$$\tau_2(s_5) = \frac{6T_0}{thb_1^3}(\frac{b_1^2}{4} - s_5^2)$$

The sign of $\tau_2(s_1)$ is positive, which means that the shear flow is in the direction of increasing s_1, hence upward. Similarly, the sign of $\tau_2(s_2)$ is negative, so the shear flow is in the direction of decreasing s_2, hence upward. Thus, warping shear stresses on the right flange are directed upward, but the signs of $\tau_2(s_4)$ and $\tau_2(s_5)$ show that the warping shear stresses on the left flange are directed downward.

The normal stress at the fixed end due to bending is

$$\sigma_1(z) = \frac{M_y z}{I_y} = -\frac{PLz}{I_y}$$

The normal warping stress is

$$\sigma_2(s) = \frac{M_\omega \omega(s)}{I_\omega}$$

where M_ω is the bimoment at the fixed end

$$M_\omega = -\bar{E}I_\omega \theta_x''(0) = -\frac{T_0}{c}\tanh cL$$

To calculate the stresses numerically, the following dimensions will be assumed

$$b_1 = b \qquad h = b_2 = 2b \qquad L = 20b \qquad t = \frac{b}{10}$$

Poisson's ratio will be taken to be $\nu = 0.25$. The modulus of elasticity $\bar{E}$ and the shear modulus G are then

$$\bar{E} = \frac{E}{1-\nu^2} = \frac{16E}{15}$$
$$G = \frac{E}{2(1+\nu)} = \frac{2E}{5}$$

The transverse and warping shear stress distributions at the clamped end of the beam are sketched in Figure 2.16. The force-couple equivalent of the transverse shear stress τ_1 at the shear center S is a single force of magnitude $V_z = P$. The warping shear stress τ_2 is statically equivalent to a couple. The total shear stress on the right flange is zero, so that, at the fixed end of the beam, all shear stresses are carried by the left flange.

The bending and warping normal stresses are shown in Figure 2.17. The maximum normal warping stress exceeds the maximum bending stress. The bending stresses are statically equivalent to the bending moment $M_y = -PL$. The warping stresses are statically equivalent to zero force and zero couple. When considered separately for the two flanges, these stresses are equivalent to two equal and opposite bending moments. The maximum stresses are shown in Table 2.1. The reference stress σ_0 is defined as

$$\sigma_0 = \frac{P}{b^2}$$

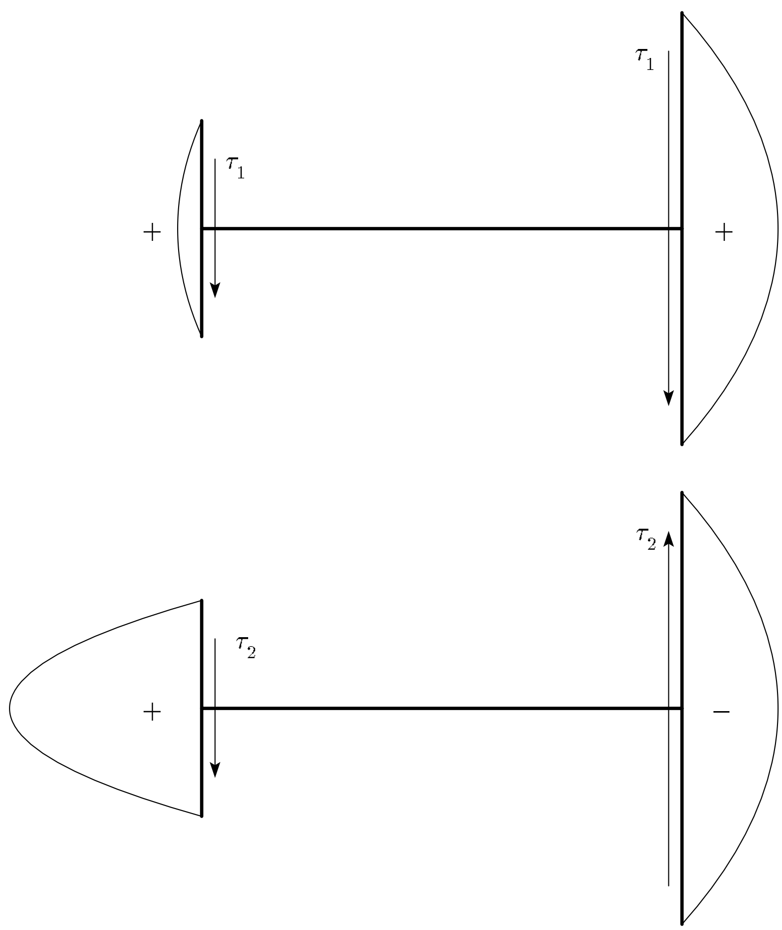

FIGURE 2.16 *Transverse and warping shear stresses at the fixed end*

As a second example, the stress distribution in the simply supported beam shown in Figure 2.18 will be determined. The load is a vertical force of magnitude P at midspan. The cross section, whose centroid C is also the shear center S, is shown in Figure 2.19. The wall thickness t is the same for the flanges and the web.

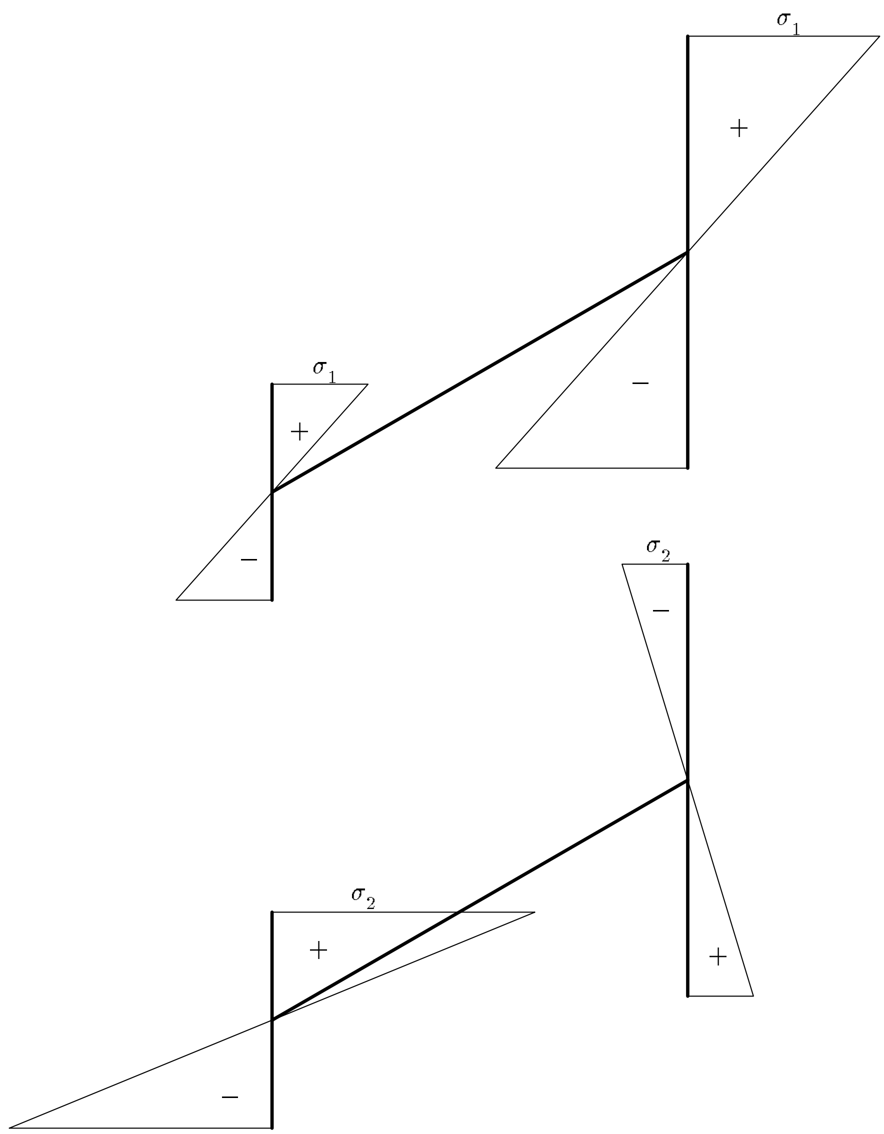

FIGURE 2.17 *Bending and warping normal stresses at the fixed end*

The area moments of inertia and the area product of inertia for this cross section are

$$I_y = \frac{2tb^3}{3} \qquad I_z = \frac{th^3}{12} + \frac{tbh^2}{2} \qquad I_{yz} = \frac{thb^2}{2}$$

The torsional constant, calculated from Saint-Venant's approximation, is

$$J = \frac{ht^3}{3} + \frac{2bt^3}{3} = \frac{t^3(h+2b)}{3}$$

MAXIMUM STRESS	RIGHT FLANGE	LEFT FLANGE
τ_1/σ_0 (Transverse)	6.7	1.7
τ_2/σ_0 (Warping)	6.7	13.3
σ_1/σ_0 (Bending)	266.7	133.3
σ_2/σ_0 (Warping)	91.3	365.0

TABLE 2.1 *Maximum stresses at the clamped end*

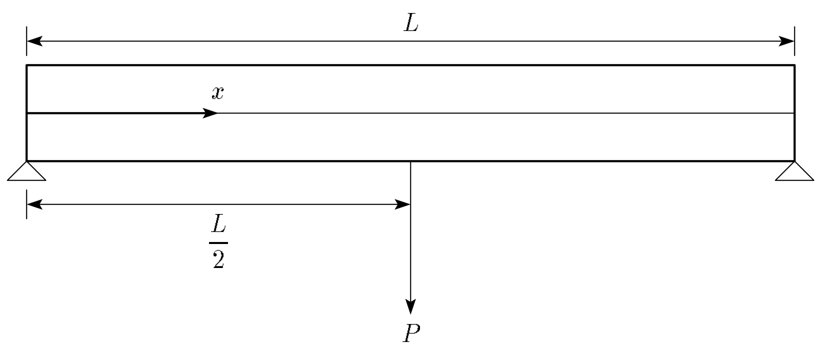

FIGURE 2.18 *Simply supported beam with midspan load*

The warping function with pole and origin both at point O is

$$\omega_O(s_1) = 0 \qquad \omega_O(s_2) = 0 \qquad \omega_O(s_3) = hs_3$$

The principal warping function is given by Eq. (2.17) as

$$\omega(s) = \omega_O(s) - \frac{Q_{\omega_O}}{A} - (y_S - y_O)z(s)$$

Since

$$Q_{\omega_O} = \int_0^b hs_3 t ds_3 = \frac{thb^2}{2}$$

and

$$A = t(h + 2b)$$

the principal warping function is

$$\omega(s_1) = \frac{hs_1}{2} - \frac{hb^2}{2(h+2b)}$$

$$\omega(s_2) = -\frac{hb^2}{2(h+2b)}$$

$$\omega(s_3) = \frac{hs_3}{2} - \frac{hb^2}{2(h+2b)}$$

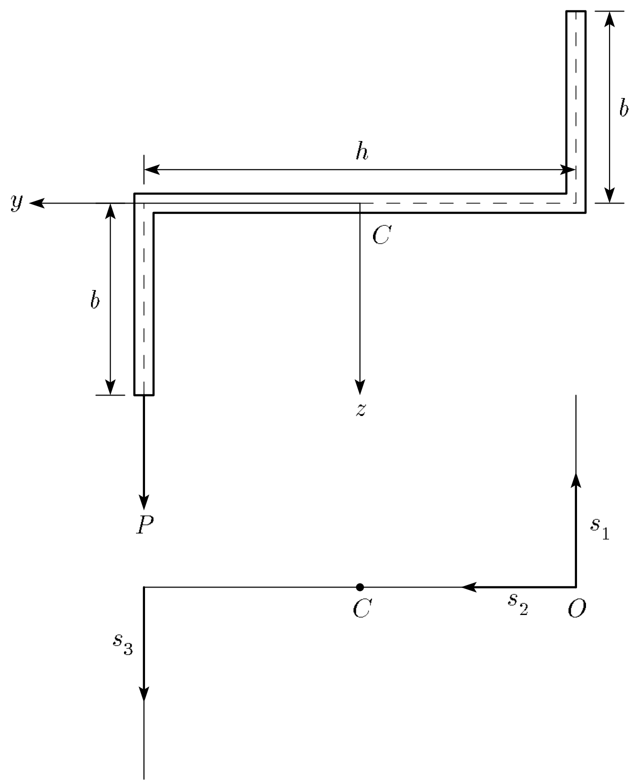

FIGURE 2.19 *Cross section of the simply supported beam*

The warping constant is found from Eq. (2.20)

$$I_\omega = I_{\omega_O} - \frac{Q^2_{\omega_O}}{A} - (y_S - y_O)^2 I_y = \frac{th^2b^3(b+2h)}{12(h+2b)}$$

The applied load at $x = L/2$ is equivalent to a torsional couple T_0 and a transverse force P at the shear center

$$T_0 = \frac{Ph}{2}$$

The applied torque per unit length can be written in terms of the Dirac delta function

$$m(x) = T_0 \delta(x - \frac{L}{2})$$

The angle of twist is calculated from Eq. (2.34) as

$$\theta_x(x) = C_1 + C_2 x + C_3 \cosh cx + C_4 \sinh cx - \phi(x)$$

where $\phi(x) = 0$ for $0 \leqslant x \leqslant L/2$ and

$$\phi(x) = \frac{T_0}{cGJ}\left[c(x - \frac{L}{2}) - \sinh c(x - \frac{L}{2})\right]$$

for $L/2 \leqslant x \leqslant L$.

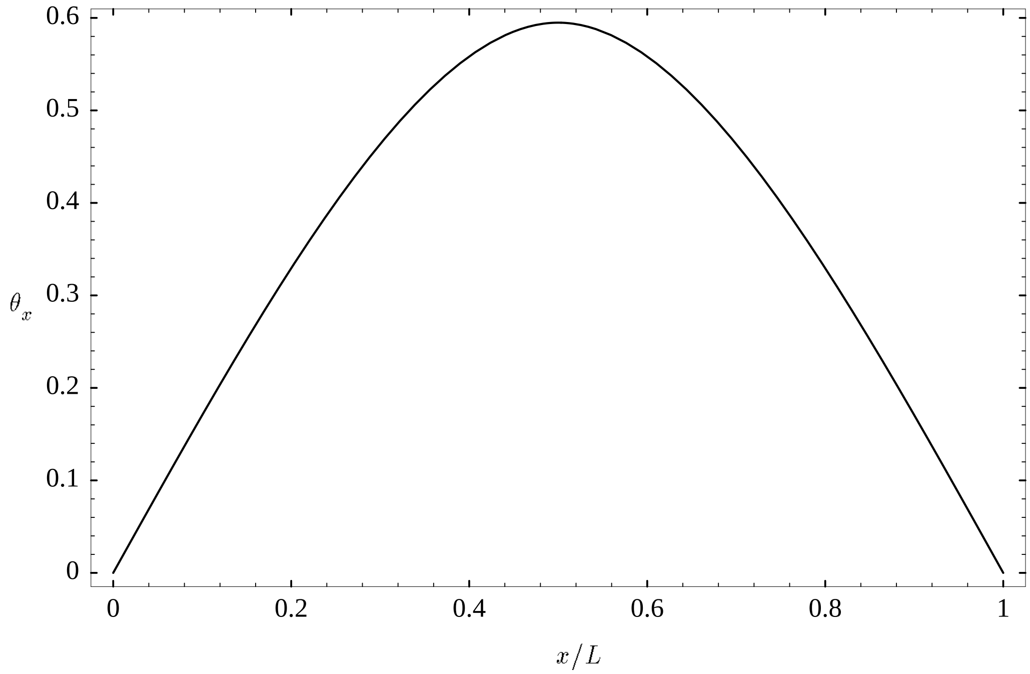

FIGURE 2.20 *Angle of twist for the simply supported beam*

The boundary conditions at the two simple supports

$$\theta_x(0) = \theta_x(L) = 0 \qquad \theta''(0) = \theta''(L) = 0$$

are solved for the integration constants

$$C_1 = 0 \qquad C_2 = \frac{T_0}{2GJ} \qquad C_3 = 0 \qquad C_4 = -\frac{T_0 \sinh cL/2}{cGJ \sinh cL}$$

For the left half of the beam

$$\theta_x(x) = \frac{T_0}{2cGJ}\left(cx - 2\frac{\sinh cL/2}{\sinh cL}\sinh cx\right)$$
$$M_\omega(x) = \frac{T_0}{c\sinh cL}\sinh\frac{cL}{2}\sinh cx$$
$$T_t(x) = \frac{T_0}{2} - \frac{T_0 \sinh cL/2}{\sinh cL}\cosh cx$$
$$T_\omega(x) = T_0\frac{\sinh cL/2}{\sinh cL}\cosh cx$$

The qualitative behavior of these functions over the entire span of the beam can be seen in Figures 2.20, 2.21, and 2.22. In Figure 2.22, the torques T_t and T_ω are shown

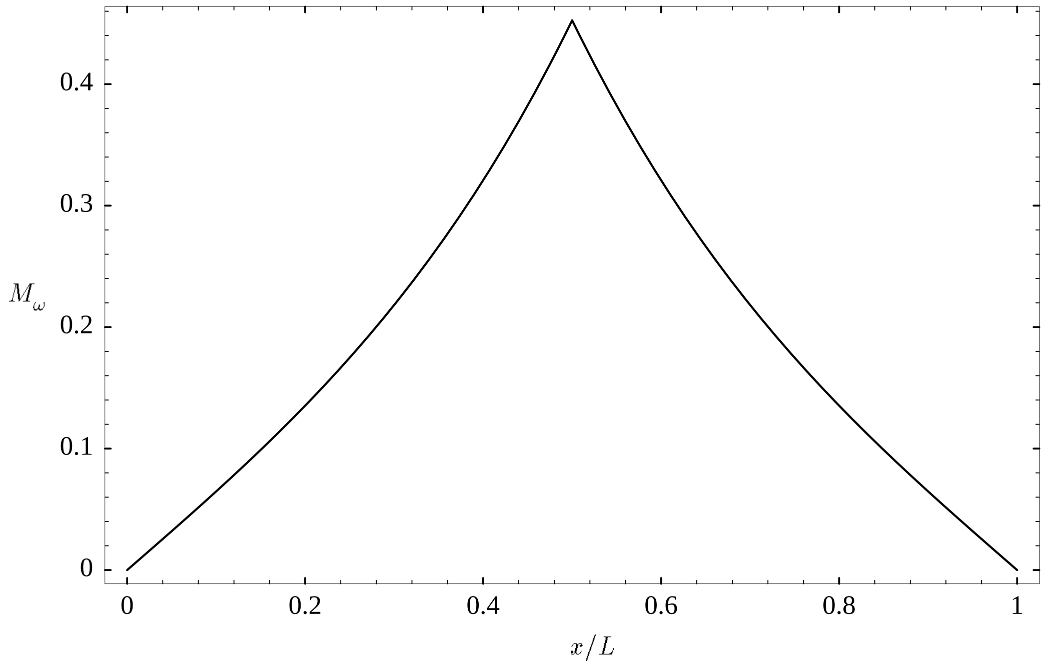

FIGURE 2.21 *Bimoment for the simply supported beam*

as fractions of the applied torque T_0. The total torque is the sum of T_t and T_ω

$$T(x) = \frac{T_0}{2} \qquad \text{if } x < \frac{L}{2}$$
$$T(x) = -\frac{T_0}{2} \qquad \text{if } x > \frac{L}{2}$$

The stresses at $x = L/2$ at the section just to the left of the applied torque will be calculated. The transverse shear stress at this section is

$$\tau_1(s) = -\frac{I_z Q_y(s) - I_{yz} Q_z(s)}{t(I_y I_z - I_{yz}^2)} V_z$$

From Figure 2.19, on the right flange, the first moments of area are

$$Q_y(s_1) = \int_b^{s_1} -s_1 t ds_1 = \frac{t(b^2 - s_1^2)}{2}$$
$$Q_z(s_1) = \int_b^{s_1} -\frac{h}{2} t ds_1 = \frac{ht(b - s_1)}{2}$$

and

$$\tau_1(s_1) = \frac{3P(s_1 - b)(bh + hs_1 + 6bs_1)}{4tb^3(2h + 3b)}$$

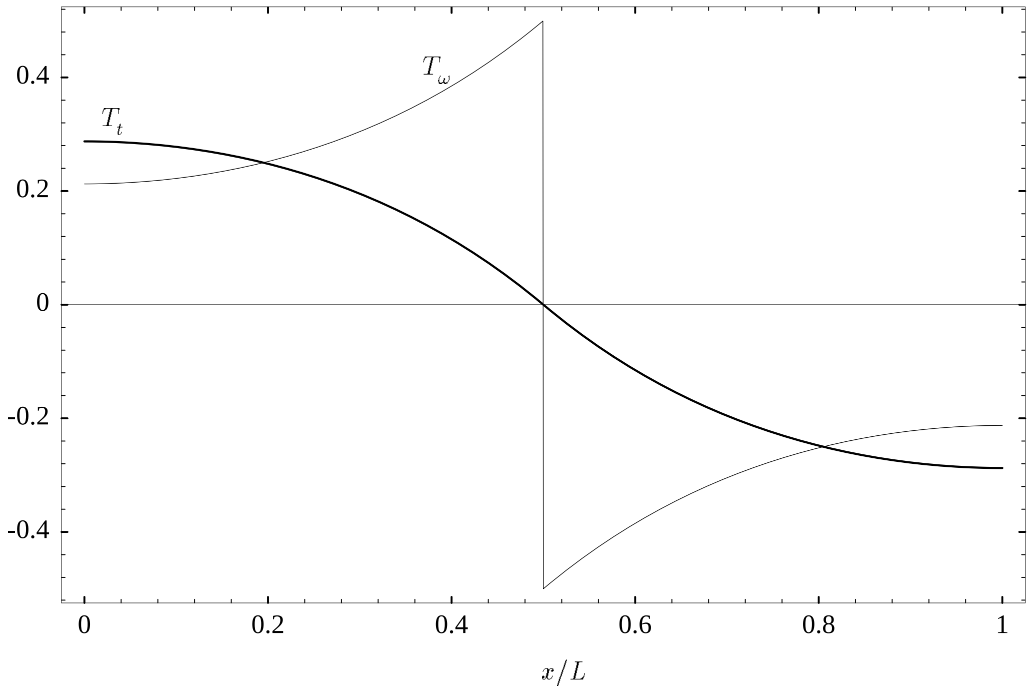

FIGURE 2.22 *Pure torsion and warping torques for the simply supported beam*

The value of $\tau_1(s_1)$ for $s_1 < b$ is negative, which means that the the stress is in the negative s_1, or the positive z, direction. For the web, the first moments of area are

$$Q_y(s_2) = -\frac{tb^2}{2}$$
$$Q_z(s_2) = -\frac{t(s_2^2 - hs_2 - bh)}{2}$$

The shear stress in the web is

$$\tau_1(s_2) = \frac{3P(6s_2^2 - 6hs_2 + h^2}{4tbh(2h + 3b)}$$

Similarly, on the left flange,

$$\tau_1(s_3) = \frac{3P(b - s_3)(bh + 6bs_3 + hs_3)}{4tb^3(2h + 3b)}$$

The warping shear stress at $x = L/2$ is

$$\tau_2(s) = -\frac{Q_\omega(s)}{tI_\omega}T_\omega$$

where the warping torque T_ω is $T_0/2$, because the pure torsion torque T_t is zero at midspan. The expressions for the warping shear stresses are

$$\tau_2(s_1) = \frac{T_0 h(b - s_1)(bh + hs_1 + 2bs_1)}{8I_\omega(h + 2b)}$$

$$\tau_2(s_2) = \frac{T_0 hb^2(2s_2 - h)}{8I_\omega(h + 2b)}$$

$$\tau_3(s_3) = \frac{T_0 h(b - s_3)(bh + hs_3 + 2bs_3)}{8I_\omega(h + 2b)}$$

The normal stress distribution at $x = L/2$ due to bending is

$$\sigma_1 = -\frac{I_{yz} M_y}{I_y I_z - I_{yz}^2} y + \frac{I_z M_y}{I_y I_z - I_{yz}^2} z$$

where $M_y = PL/4$. The normal stress due to warping is

$$\sigma_2 = \frac{M_\omega \omega}{I_\omega}$$

where

$$M_\omega = \frac{T_0}{2c} \tanh \frac{cL}{2}$$

The shear stress distribution at $x = L/2$ is sketched in Figure 2.23. The force resultant of the transverse shear stress τ_1 over the two flanges is equal to the total shear force $P/2$. The transverse shear stresses over the web are statically equivalent to a zero force-couple. The warping shear stress τ_2 is equivalent a torsional moment. The transverse shear stress adds to the warping shear stress over the left flange, but subtracts from it over the right flange.

The normal stress distribution at $x = L/2$ is sketched in Figure 2.24. The normal stress σ_1 due to bending is statically equivalent to a bending moment about the y axis. The warping stress σ_2 is statically equivalent to a zero force-couple. The bending and warping stresses are additive over the left flange.

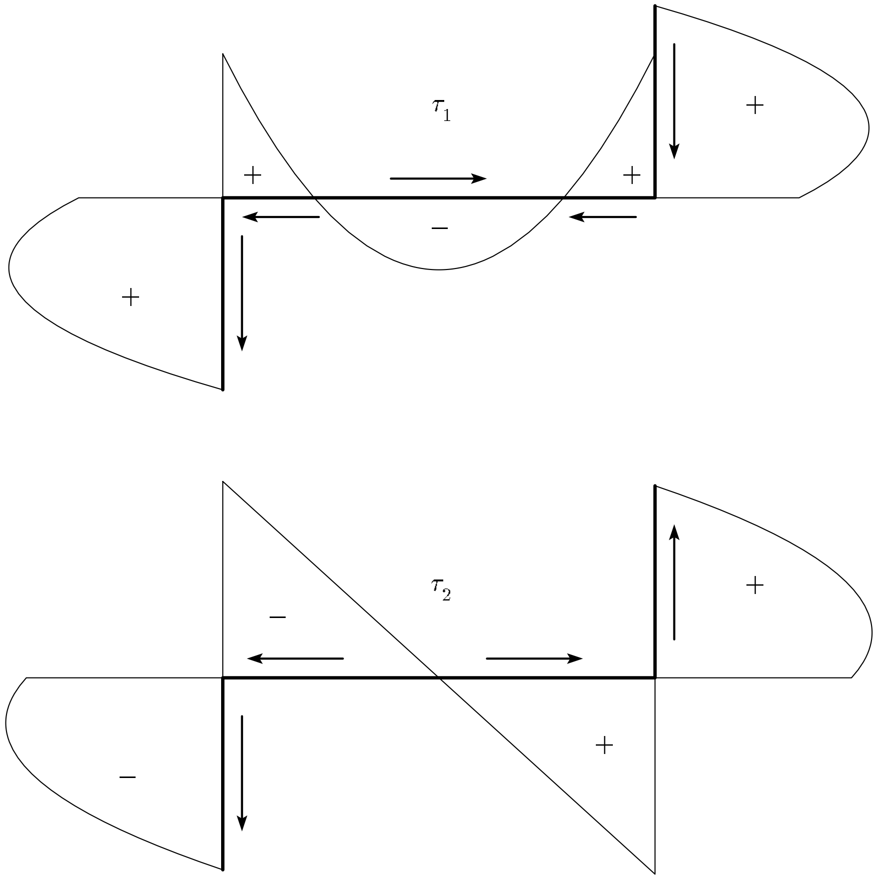

FIGURE 2.23 *Shear stresses for the simply supported beam*

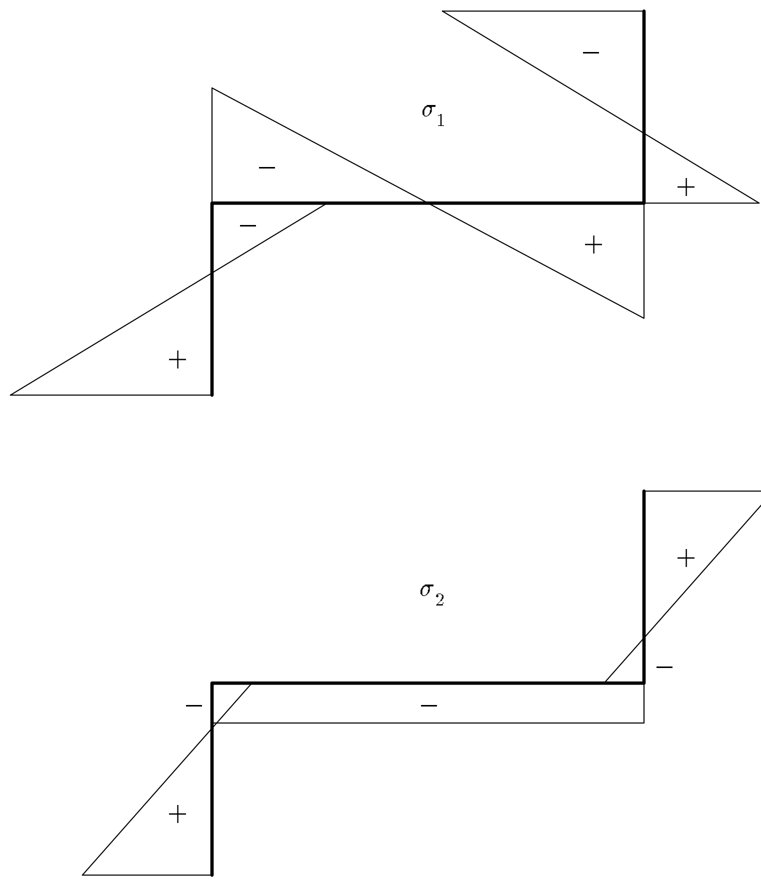

FIGURE 2.24 *Normal stresses for the simply supported beam*

CHAPTER III

THIN-WALLED ELASTIC BEAMS OF CLOSED CROSS SECTION

3.1. Geometry of Deformation

A closed thin-walled cross section is shown in Figure 3.1. The tangential and normal coordinates, s and n, are chosen so that the axes n, s, x form a right-handed triad. The coordinate s traces the median line starting from an arbitrarily selected origin, and the y, z coordinates of any point on the median line are functions of s. The normal coordinate n of any point of the median line is zero. The angle $\alpha(s)$ is measured from the positive y axis to the positive n axis.

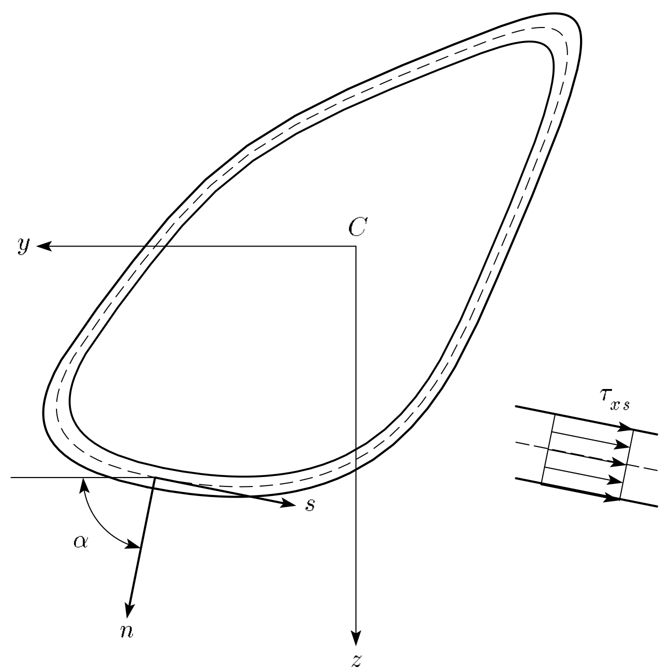

FIGURE 3.1 *Closed thin-walled section*

As in Chapter II, it will be assumed that the shape of the median line and its dimensions remain unchanged in the yz plane when the beam undergoes a deformation under static loads. This means that the transverse displacements, which are defined as the displacement components in the plane of the undeformed cross section, of a point on the median line are those of a point belonging to a plane rigid curve constrained to move in its own plane. Let S be the shear center of the cross section shown in Figure 3.2 and let $\eta(s)$ denote the tangential component of the point of the median line at the coordinate s. As shown in Chapter II, $\eta(s)$ can be written as

$$\eta(x,\, s) = v_S(x)\cos\beta(s) + w_S(x)\sin\beta(s) + \theta_x(x)h(s) \tag{3.1}$$

where v_S, w_S are the displacements of the shear center in the y, z directions, and h is the projection, onto the unit normal vector $\mathbf{e}_n$, of the position vector $\mathbf{r}(s)$ of the point at s

$$h = \mathbf{r} \cdot \mathbf{e}_n$$

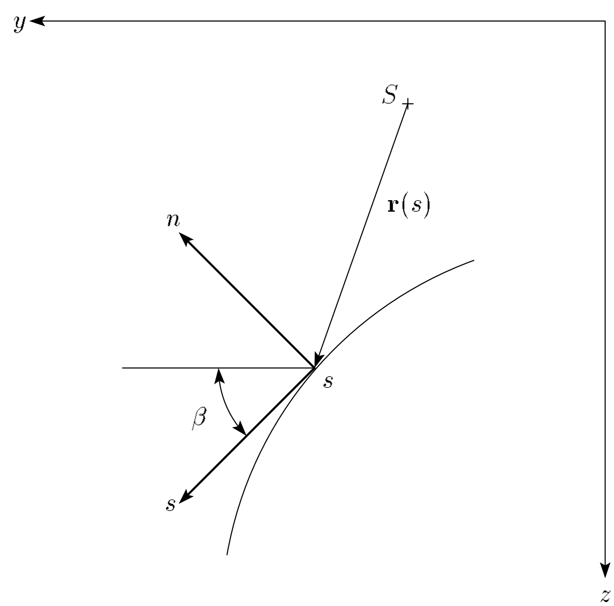

FIGURE 3.2 *Tangential and normal components of displacement*

It will now be assumed that the shear strain γ_{xs} of the median line is equal to its value found in Saint-Venant torsion. This assumption can be written as

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \eta}{\partial x} = \frac{\tau_{xs}}{G} = \frac{q_t}{tG}$$

where q_t is the constant shear flow of Saint-Venant torsion

$$q_t = \frac{T_t}{\Omega}$$

and Ω denotes twice the area enclosed by the median line

$$\Omega = \oint h ds$$

The derivative of u with respect to s is

$$\frac{\partial u}{\partial s} = \frac{T_t}{tG\Omega} - v'_S \cos\beta - w'_S \sin\beta - \theta'_x h$$

from which the displacement of the point at s along the x axis is obtained by integration

$$u = u_0 + \int_0^s \frac{T_t}{tG\Omega} ds - \theta'_x \int_0^s h ds - v'_S y - w'_S z$$

From Eq. (1.18)

$$\frac{T_t}{G\Omega} = \frac{\theta'_x \Omega}{\oint \frac{ds}{t(s)}}$$

with which the axial displacement becomes

$$u = u_0 + \theta'_x \frac{\Omega}{\oint \frac{ds}{t}} \int_0^s \frac{ds}{t} - \theta'_x \int_0^s h ds - v'_S y - w'_S z$$

The *warping function* for a closed section is defined by

$$\omega(s) = \int_0^s h ds - \frac{\Omega}{\oint \frac{ds}{t}} \int_0^s \frac{ds}{t} \tag{3.2}$$

The first term of the preceding equation will be recognized as the sectorial area, or the warping function for an open section. The warping displacement can now be written in the same form as it was in Chapter II for open cross sections

$$u(x,\, s) = u_0(x) - v'_S(x)y(s) - w'_S(x)z(s) - \theta'_x(x)\omega(s) \tag{3.3}$$

It is easily verified that the presence of the second integral in Eq. (3.2) does not change Eq. (2.7) for changing the pole of the warping function from B to A

$$\omega_A(s) = \omega_B(s) - \omega_B(s_0) + (z_A - z_B)(y(s) - y_0) - (y_A - y_B)(z(s) - z_0) \tag{3.4}$$

The equations for finding the principal pole, or the shear center, also remain the same as those for open sections

$$y_S = y_B + \frac{I_{z\omega_B} I_z - I_{y\omega_B} I_{yz}}{I_y I_z - I_{yz}^2} \tag{3.5}$$

$$z_S = z_B + \frac{I_{z\omega_B} I_{yz} - I_{y\omega_B} I_y}{I_y I_z - I_{yz}^2} \tag{3.6}$$

The principal warping function is given in terms of a warping function ω_B, whose pole and origin are arbitrarily chosen, by Eq. (2.17)

$$\omega(s) = \omega_B(s) - \frac{Q_{\omega_B}}{A} + (z_S - z_B)y(s) - (y_S - y_B)z(s) \tag{3.7}$$

Similarly, the warping constant is given by

$$I_\omega = I_{\omega_B} - \frac{Q_{\omega_B}^2}{A} - (y_S - y_B)^2 I_y + 2(y_S - y_B)(z_S - z_B) I_{yz} - (z_S - z_B)^2 I_z \tag{3.8}$$

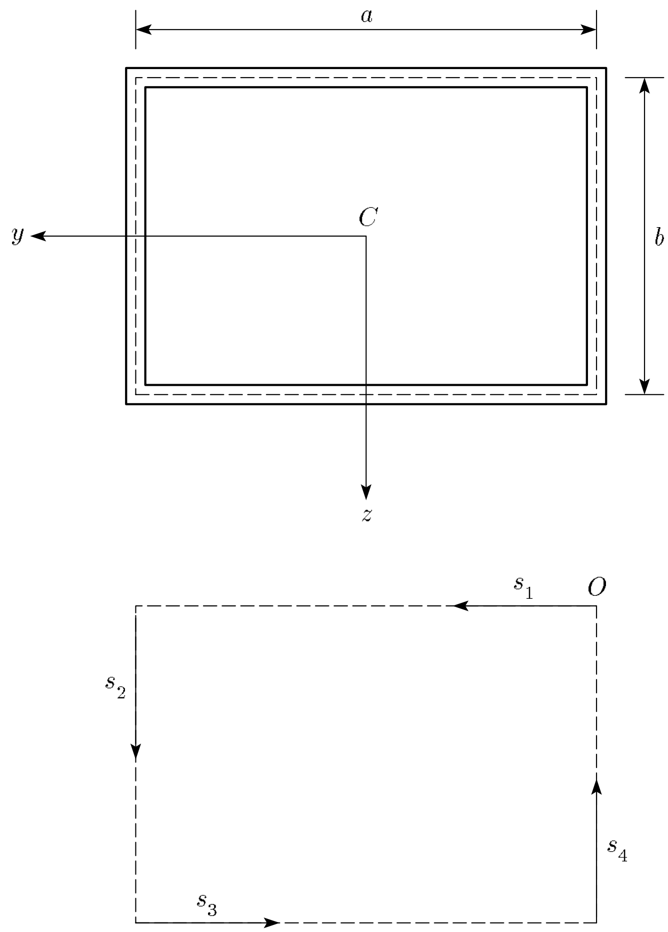

FIGURE 3.3 *Thin-walled box section*

As an example, consider the thin-walled rectangular cross section of uniform thickness t shown in Figure 3.3. If point O is used both as pole and origin, the warping function is

$$\omega_O(s_1) = -\frac{ab}{a+b}s_1$$
$$\omega_O(s_2) = \frac{a^2}{a+b}(s_2 - b)$$
$$\omega_O(s_3) = \frac{b^2}{a+b}s_3$$
$$\omega_O(s_4) = \frac{ab}{a+b}(b - s_4)$$

The area moments of inertia are

$$I_y = \frac{tb^2(b+3a)}{6} \qquad I_z = \frac{ta^2(a+3b)}{6}$$

The shear center is at the centroid of the rectangle. The principal warping function is found by an application of Eq. (3.7)

$$\omega(s_1) = \frac{b(b-a)}{4(a+b)}(2s_1 - a)$$
$$\omega(s_2) = \frac{a(a-b)}{4(a+b)}(2s_2 - b)$$
$$\omega(s_3) = \frac{b(b-a)}{4(a+b)}(2s_3 - a)$$
$$\omega(s_4) = \frac{a(a-b)}{4(a+b)}(2s_4 - b)$$

The principal warping function is zero for a square cross section, which according to the theory being described here, is free of warping. The warping constant for the rectangular box section is

$$I_\omega = \frac{ta^2b^2(b-a)^2}{24(a+b)}$$

In the theory developed by Benscoter for closed thin-walled sections, the rate of angle of twist θ'_x in Eq. (3.3) is replaced by an arbitrary function ϑ of x, so that the fundamental kinematical assumption for the warping displacement becomes

$$u(x,\, s) = u_0(x) - v'_S(x)y(s) - w'_S(x)z(s) - \vartheta(x)\omega(s) \tag{3.9}$$

The normal strain ϵ_x is written as

$$\epsilon_x = u'_0 - v''_S y - w''_S z - \vartheta'\omega \tag{3.10}$$

The shear strain γ_{xs} is

$$\gamma_{xs} = \frac{\partial u}{\partial s} + \frac{\partial \eta}{\partial x} = \theta'_x h - \vartheta\frac{\partial \omega}{\partial s} = \theta'_x h - \vartheta\left(h - \frac{\Omega}{t\oint\frac{ds}{t}}\right) \tag{3.11}$$

3.2. Equations of Equilibrium

The normal stress σ_x is obtained from Hooke's law as

$$\sigma_x = \bar{E}\epsilon_x = \bar{E}(u'_0 - v''_S y - w''_S z - \vartheta'\omega)$$

The shear stress is the sum of the bending and the torsional contributions

$$\tau_{xs} = \tau_b + \tau_t$$

The torsional contribution is

$$\tau_t = G\gamma_{xs} = G\theta'_x h - G\vartheta(h-k)$$

where the abbreviation

$$k = \frac{\Omega}{t\oint\frac{ds}{t}}$$

has been introduced. As for open cross sections, the bending shear stress has no corresponding shear strain.

The stress resultants for the normal stress σ_x are the axial force N, the bending moments M_y and M_z, and the *bimoment* M_ω, which are defined by

$$N = \int \sigma_x dA$$
$$M_y = \int z\sigma dA$$
$$M_z = -\int y\sigma dA$$
$$M_\omega = \int \omega\sigma dA$$

The stress resultants are evaluated, recalling that the origin of the coordinates y, z is the centroid, and that ω is the principal warping function

$$\begin{pmatrix} N \\ M_y \\ M_z \\ M_\omega \end{pmatrix} = \bar{E} \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & -I_{yz} & -I_y & 0 \\ 0 & I_z & I_{yz} & 0 \\ 0 & 0 & 0 & -I_\omega \end{pmatrix} \begin{pmatrix} u_0' \\ v_S'' \\ w_S'' \\ \vartheta_x' \end{pmatrix}$$

Hence

$$\bar{E}u_0'(x) = \frac{N(x)}{A}$$
$$\bar{E}v_S''(x) = \frac{I_{yz}M_y(x) + I_y M_z(x)}{I_y I_z - I_{yz}^2}$$
$$\bar{E}w_S''(x) = -\frac{I_z M_y(x) + I_{yz} M_z(x)}{I_y I_z - I_{yz}^2}$$
$$\bar{E}\vartheta_x'(x) = -\frac{M_\omega(x)}{I_\omega}$$

The normal stress is found in terms of the stress resultants by using these expressions in Eq. (2.21)

$$\sigma_x = \frac{N}{A} - \frac{I_{yz}M_y + I_y M_z}{I_y I_z - I_{yz}^2} y + \frac{I_z M_y + I_{yz} M_z}{I_y I_z - I_{yz}^2} z + \frac{M_\omega \omega}{I_\omega} \tag{3.12}$$

The total torque T is

$$T = \int h\tau_{xs} dA = G\theta_x' \int h^2 dA - G\vartheta \left(\int h^2 dA - \int hk dA \right)$$

The last part of this expression can be evaluated as

$$\int hk dA = \int \frac{\Omega h dA}{t \oint \frac{ds}{t}} = \frac{\Omega^2}{\oint \frac{ds}{t}} = J$$

A cross-sectional property, sometimes called the *polar constant*, is defined by

$$I_h = \int h^2 dA$$

so that the torque T is

$$T = GI_h\theta_x' - G\vartheta(I_h - J)$$

The torque equilibrium equation for a length dx of the beam gives

$$\frac{dT}{dx} + m(x) = GI_h\theta_x'' - G\vartheta'(I_h - J) + m(x) = 0$$

where $m(x)$ is the applied torque about the shear center S per unit length of the beam.

As in Chapter II, the equilibrium equation in the longitudinal direction, in the absence of applied axial load p_x, gives

$$t\frac{\partial \sigma_x}{\partial x} + \frac{\partial q}{\partial s} = 0$$

The equilibrium of forces in the longitudinal direction remains the same

$$\bar{E}u_0''(x)A + p_x = 0$$

which gives $u_0''(x) = 0$. Hence

$$\frac{\partial q}{\partial s} = -t\frac{\partial \sigma_x}{\partial x} = t\bar{E}(v_S'''y + w_S'''z + \vartheta''\omega)$$

The shear flow is

$$q(x,\, s) = q_0(x) + \bar{E}\big(v_S'''(x)Q_z(s) + w_S'''(x)Q_y(s) + \vartheta_x''(x)Q_\omega(s)\big) \tag{3.13}$$

As in Chapter II, the shear stress resultants V_y, V_z, and T_ω be defined by

$$V_y = \int q(x,\, s)dy$$

$$V_z = \int q(x,\, s)dz$$

$$T_\omega = \int q(x,\, s)d\omega$$

The warping torque is calculated by integrating both sides of Eq. (3.13) with respect to ω

$$T_\omega = \bar{E}\vartheta_x''Q_\omega(s)d\omega = -\bar{E}\vartheta_x''I_\omega$$

because, as shown Chapter II by an integration by parts,

$$\int Q_\omega(s)d\omega = -I_\omega$$

Since

$$\int q(h-k)ds = G(I_h - J)(\theta_x' - \vartheta)$$

it follows that

$$-\bar{E}I_\omega\vartheta''' = G(I_h - J)(\theta_x'' - \vartheta')$$

The equation for the angle of twist is found by eliminating ϑ from

$$GI_h\theta_x'' - G\vartheta'(I_h - J) + m(x) = 0$$

$$G(I_h - J)(\theta_x'' - \vartheta') + \bar{E}I_\omega\vartheta''' = 0$$

From the first of these equations

$$\vartheta' = \frac{I_h}{I_h - J}\theta_x'' + \frac{m}{G(I_h - J)}$$

The differential equation for the angle of twist is

$$\frac{\bar{E}I_\omega I_h}{I_h - J}\theta_x^{iv} - GJ\theta_x'' = m(x) - \frac{\bar{E}I_\omega}{G(I_h - J)}m''(x) \tag{3.14}$$

When attempting to use Eq. (3.14), it is possible to encounter cross sections for which I_h and J are equal. For the rectangular box section of Figure 3.3, the polar constant I_h is

$$I_h = \frac{tab(a+b)}{2}$$

and the torsional constant J is

$$J = \frac{2ta^2b^2}{a+b}$$

For a square cross section, with $a = b$,

$$I_h = J = tb^3$$

and Eq. (3.14) cannot be used. In general, when the polar constant is the identical to the torsional constant, the cross section is free of warping, and the warping function is everywhere zero. The differential equation for the angle of twist then reduces to

$$GJ\theta_x'' + m(x) = 0$$

This is the governing equation for Saint-Venant torsion with a variable distributed moment $m(x)$.

3.3. A Multicell Analysis Example

If the area enclosed by the outer wall of a cross section is subdivided into any number of other closed thin-walled sections, the beam is a *multicell* structure, an example of which is shown in Figure 3.4. For such cross sections, the condition obtained in Chapter I by taking the line integral of the derivative of the longitudinal displacement u with respect to s around a closed contour is used

$$\frac{1}{G}\oint_i q\frac{ds}{t(s)} - \theta_x'\Omega_i = 0 \tag{3.15}$$

where the integral is taken around the contour of the ith cell, and Ω_i is twice the area enclosed by the contour of the ith cell.

In pure torsion the shear flow in each cell has a constant value. On a shared wall, such as the one of length b in Figure 3.4, the shear flows are additive

$$q_{12} = q_2 - q_1$$

where q_1, q_2 are the individual shear flows in the two cells and q_{12} is the shear flow in the shared part of the wall. The condition in Eq. (3.15) is applied to the two cells, assuming that the thickness t is uniform throughout the cross section,

$$\theta_x'\Omega_1 - \frac{2q_1(d+e)}{Gt} + \frac{q_2b}{Gt} = 0$$
$$\theta_x'\Omega_2 + \frac{q_1b}{Gt} - \frac{2q_2(a+b)}{Gt} = 0$$

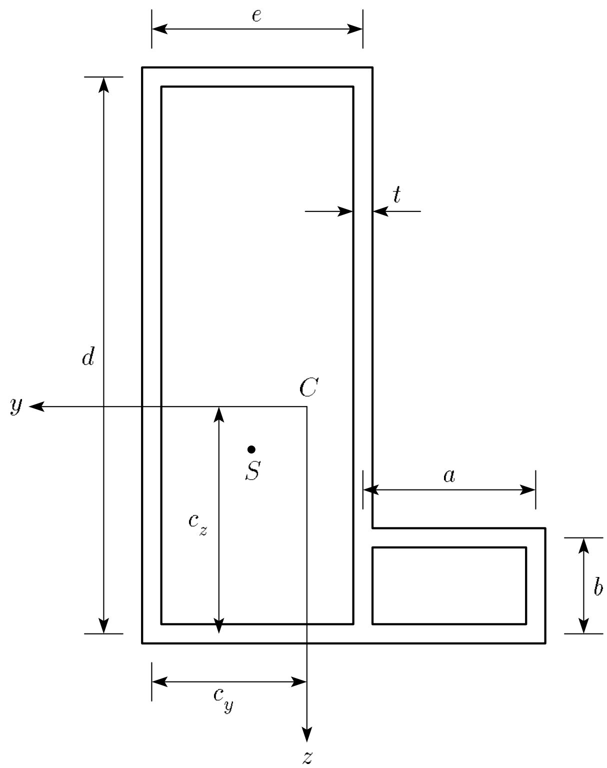

FIGURE 3.4 *Two-cell thin-walled cross section*

The directions assumed for the shear flows q_1, q_2 are indicated in Figure 3.5. The total torque in the section is

$$T = q_1\Omega_1 + q_2\Omega_2$$

The torsional constant can be calculated from

$$J = \frac{T}{G\theta'_x} = \frac{q_1\Omega_1 + q_2\Omega_2}{G\theta'_x}$$

The warping function with respect to any arbitrarily chosen pole O is written as

$$\omega_O(s) = \int_0^s h ds - \frac{1}{G}\int_0^s \overline{q}\frac{ds}{t}$$

where the first integral is the sectorial area. In the second integral, $\overline{q}$ denotes the shear flow corresponding to $\theta'_x = 1$. For the example two-cell section, with the s coordinates defined in Figure 3.5, the warping function ω_O, whose pole and origin

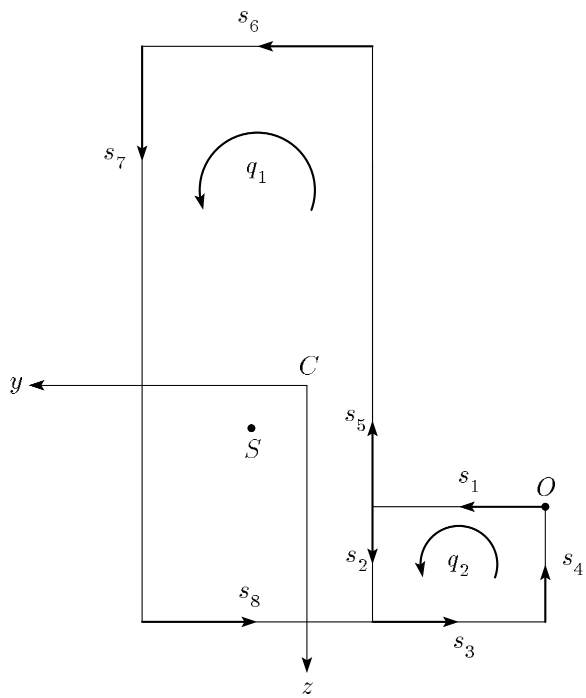

FIGURE 3.5 *Definitions for the two-cell thin-walled section*

are both chosen to be point O, is written as

$$\begin{aligned}
\omega_O^{(1)}(s_1) &= -\frac{\overline{q}_2 s_1}{Gt} \\
\omega_O^{(2)}(s_2) &= \omega_O^{(1)}(a) + a s_2 - \frac{(\overline{q}_2 - \overline{q}_1) s_2}{Gt} \\
\omega_O^{(3)}(s_3) &= \omega_O^{(2)}(b) + b s_3 - \frac{\overline{q}_2 s_3}{Gt} \\
\omega_O^{(4)}(s_4) &= \omega_O^{(3)}(a) - \frac{\overline{q}_2 s_4}{Gt} \\
\omega_O^{(5)}(s_5) &= \omega_O^{(1)}(a) - a s_5 - \frac{\overline{q}_1 s_5}{Gt} \\
\omega_O^{(6)}(s_6) &= \omega_O^{(5)}(d-b) + (d-b) s_6 - \frac{\overline{q}_1 s_6}{Gt} \\
\omega_O^{(7)}(s_7) &= \omega_O^{(6)}(e) + (a+e) s_7 - \frac{\overline{q}_1 s_7}{Gt} \\
\omega_O^{(8)}(s_8) &= \omega_O^{(7)}(d) + b s_8 - \frac{\overline{q}_1 s_8}{Gt}
\end{aligned}$$

Let the dimensions of the cross section be

$$d = 100 \text{ mm} \qquad e = 40 \text{ mm} \qquad b = 20 \text{ mm} \qquad a = 30 \text{ mm} \qquad t = 0.25 \text{ mm}$$

Then, based on centerline dimensions, the location of the centroid C is defined by

$$c_y = 28.61 \text{ mm} \qquad c_z = 41.11 \text{ mm}$$

The area moments of inertia and the area product of inertia, based on centerline dimensions, are found to be

$$I_y = 118222 \text{ mm}^4 \qquad I_z = 47993 \text{ mm}^4 \qquad I_{yz} = -24111 \text{ mm}^4$$

The sectorial areas

$$\Omega_1 = 2ed = 8000 \text{ mm}^2 \qquad \Omega_2 = 2ab = 1200 \text{ mm}^2$$

are needed in the calculation of the torsional constant J. The shear flows are

$$q_1 = \frac{515G\theta'_x}{69} \qquad q_2 = \frac{310G\theta'_x}{69}$$

where the numerical values in the numerators are in mm^2. With the shear flows determined, the torsional constant can be calculated

$$J = \frac{q_1\Omega_1 + q_2\Omega_2}{G\theta'_x} = 65101 \text{ mm}^4$$

and the warping function ω_O becomes

$$\begin{aligned}
\omega_O(s_1) &= -\frac{1240s_1}{69} \\
\omega_O(s_2) &= \frac{10(289s_2 - 3720)}{69} \\
\omega_O(s_3) &= \frac{20(7s_3 + 1030)}{69} \\
\omega_O(s_4) &= \frac{1240(20 - s_4)}{69} \\
\omega_O(s_5) &= -\frac{10(3720 + 413s_5)}{69} \\
\omega_O(s_6) &= \frac{20(173s_6 - 18380)}{69} \\
\omega_O(s_7) &= \frac{10(277s_7 - 22920)}{69} \\
\omega_O(s_8) &= \frac{40(1195 - 17s_8)}{69}
\end{aligned}$$

where the dimension of the s coordinates is in mm, and the dimension of ω_O is in mm^2.

Next the section properties dependent on ω_O are calculated

$$Q_{\omega_O} = \int \omega_O dA = 129076 \text{ mm}^4$$
$$I_{\omega_O} = \int \omega_O^2 dA = 482.161\,(10^6) \text{ mm}^6$$
$$I_{y\omega_O} = \int y\omega_O dA = -575148 \text{ mm}^5$$
$$I_{z\omega_O} = \int z\omega_O dA = 5703540 \text{ mm}^5$$

The shear center coordinates are

$$y_S = y_O + \frac{I_{z\omega_O} I_z - I_{y\omega_O} I_{yz}}{I_y I_z - I_{yz}^2} = 9.64 \text{ mm}$$
$$z_S = y_O + \frac{I_{z\omega_O} I_{yz} - I_{y\omega_O} I_y}{I_y I_z - I_{yz}^2} = 7.46 \text{ mm}$$

The principal warping function, whose pole is the shear center S, can be obtained from ω_O by the transformation

$$\omega(s) = \omega_O(s) - \frac{Q_{\omega_O}}{A} + (z_S - z_O)y(s) - (y_S - y_O)z(s)$$

where A is the cross-sectional area

$$A = t(2e + 2d + b + 2a) = 90 \text{ mm}^2$$

The warping constant I_ω can be determined either by integrating the square of the principal warping function over the cross-sectional area, or by the transformation formula

$$\begin{aligned} I_\omega &= I_{\omega_O} - \frac{Q_{\omega_O}^2}{A} - (y_S - y_O)I_y - (z_S - z_O)I_z + 2(y_S - y_O)(z_S - z_O)I_{yz} \\ &= 13.8514\,(10^6) \text{ mm}^6 \end{aligned}$$

3.4. Cross Sections with Open and Closed Parts

Some cross sections contain both closed cells and open branches. An example is shown in Figure 3.7. In analyzing such cross sections, the warping functions for the open branches are found as described in Chapter II. The warping functions for the closed cells are found as described in the two-cell example of the preceding section. The contribution of the open branches to the torsional constant J is usually negligibly small, so that J can be calculated for the closed cells of the cross section alone.

For the cross section shown in Figure 3.7, the origin of the user coordinate system is placed at point O, with the y axis horizontal and pointing left, the z axis vertically downward. The node coordinates are shown in Table 3.1 below. The wall thicknesses are in Table 3.2, each line entry of which lists two nodes and the thickness of the wall segment between them.

The properties of the cross section are listed in Table 3.3. A qualitative idea of the distributions of normal stress and strain due to warping is provided by the principal warping function. Because this section has straight walls, the warping

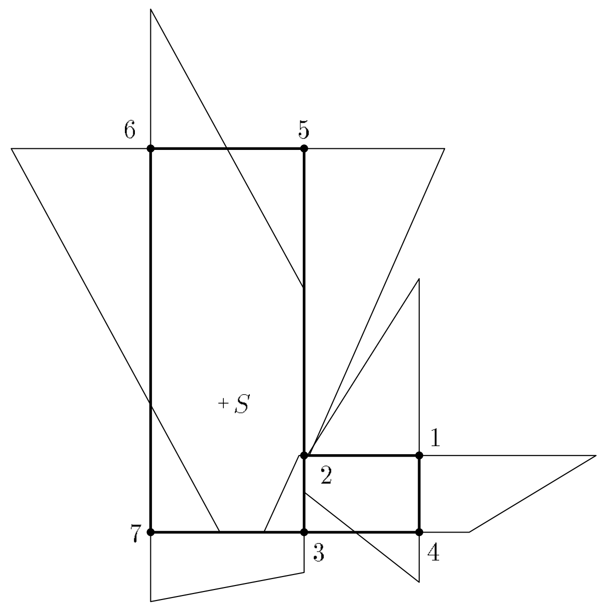

FIGURE 3.6 *Principal warping function for the two-cell thin-walled section*

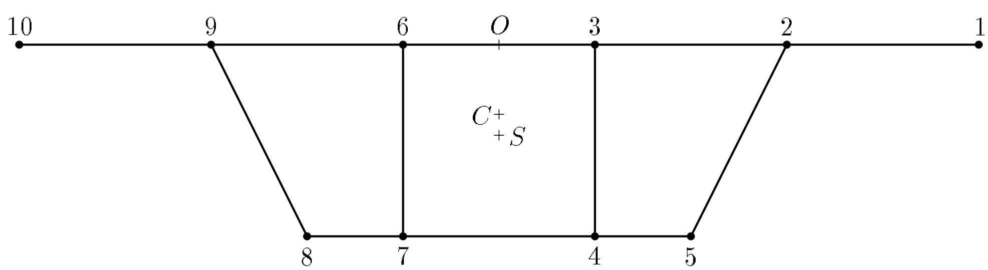

FIGURE 3.7 *Thin-walled section with three cells*

function is piecewise linear. Table 3.4 lists numerical values of the warping function, and Figure 3.8 shows how it varies along the median line.

As a final example, the cross section shown in Figure 3.9, which resembles certain thin-walled sections found in automobile frames, will be analyzed. The user coordinate system for this cross section has its origin at point O, with the y axis horizontal and directed toward the left and the z axis vertically downward. The position coordinates of the nodes of the section are listed in Table 3.5. The thickness is uniform for the entire cross section. The principal warping function for the section is sketched in Figure 3.10.

The results derived above by the approximate linear theory of beams with thin-walled cross section are compared with the results calculated by the the computer program BEAMSTRESS in Table 3.7. When the wall thickness is very small,

Node	y coordinate (mm)	z coordinate (mm)
1	-250	0
2	-150	0
3	-50	0
4	-50	100
5	-100	100
6	50	0
7	50	100
8	100	100
9	150	0
10	250	0

TABLE 3.1 *Nodal coordinates of the cross section shown in Figure 3.7*

First Node	Second Node	Thickness (mm)
1	2	10
2	3	10
3	6	10
6	9	10
9	10	10
8	9	5
7	6	5
4	3	5
5	2	5
8	7	12
7	4	12
4	5	12

TABLE 3.2 *Wall thicknesses of the cross section shown in Figure 3.7*

the linear theory agrees well with the results obtained by this program, which provides a finite-element calculation based on the elasticity formulation. As the wall thickness increases, the linear theory results become less accurate, and it is possible to have very large errors in the section properties, especially in the warping constant. As the thickness is changed, the warping function of the linear theory does not change, because the median line of the section determines this function. The elasticity formulation considers the warping function as a function of y and z, and when the boundary of the section is changed, the warping function also changes. The large errors in I_{yz} are a consequence of the assumption in the linear theory that the elements of cross-sectional area are entirely concentrated at the median line.

Cross-Sectional Area (mm^2)	9518.0
Centroid y_C (User Coordinates) (mm)	0
Centroid z_C (User Coordinates) (mm)	36.34
Shear Center y_S (Centroidal Coordinates) (mm)	0
Shear Center z_S (Centroidal Coordinates) (mm)	10.90
Shear Center y_S (User Coordinates) (mm)	0
Shear Center z_S (User Coordinates) (mm)	47.24
Area Moment of Inertia I_y (mm^4)	$18.49\ 10^6$
Area Moment of Inertia I_z (mm^4)	$132.37\ 10^6$
Area Product of Inertia I_{yz} (mm^4)	0
Polar Constant I_h (mm^4)	$34.63\ 10^6$
Torsional Constant J (mm^4)	$29.17\ 10^6$
Warping Constant I_ω (mm^6)	$10.41\ 10^9$

TABLE 3.3 *Properties of the cross section in Figure 3.7*

Node m	Node n	ω_m (mm^2)	ω_n (mm^2)
2	3	1483.29	1102.45
3	4	1102.45	−249.421
4	5	−249.421	261.184
5	2	261.184	1483.29
3	6	1102.45	−1102.45
6	7	−1102.45	249.421
7	4	249.421	−249.421
6	9	−1102.45	−1483.29
9	8	−1483.29	−261.184
8	7	−261.184	249.421
1	2	−3241.12	1483.29
9	10	−1483.29	3241.12

TABLE 3.4 *Values of the principal warping function for the section in Figure 3.7*

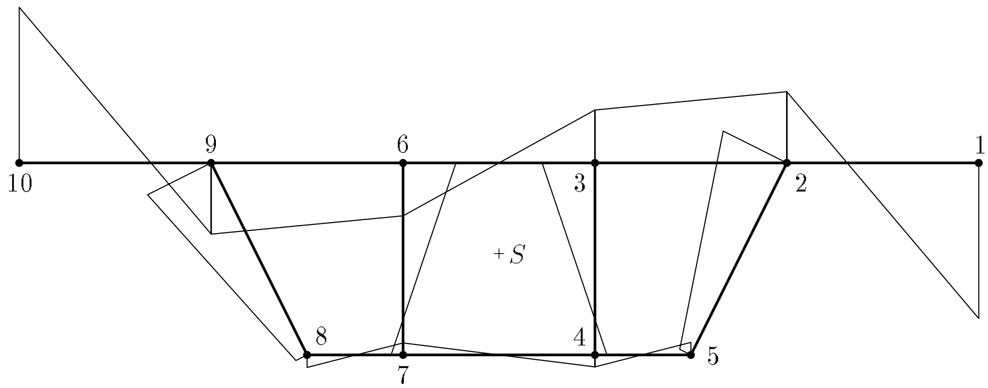

FIGURE 3.8 *Warping function for the section in Figure 3.7*

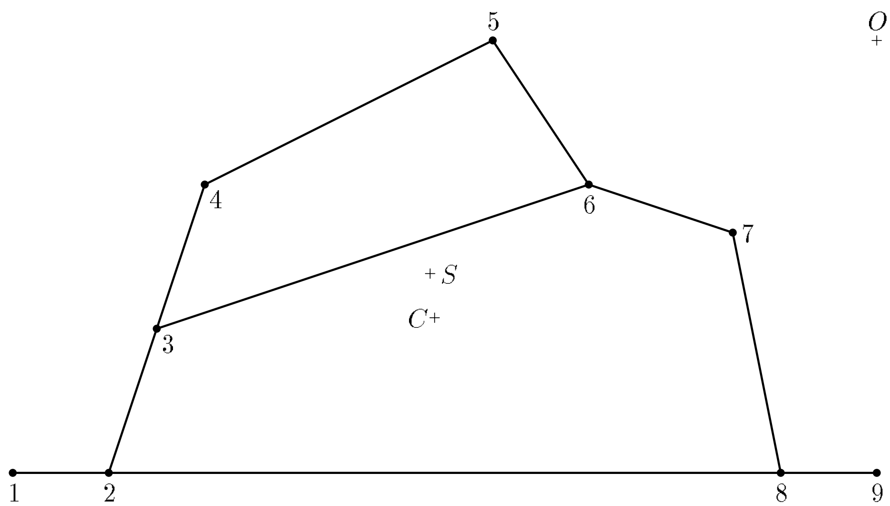

FIGURE 3.9 *Thin-walled cross section*

Node	y coordinate (mm)	z coordinate (mm)
1	180	90
2	160	90
3	150	60
4	140	30
5	80	0
6	60	30
7	30	40
8	20	90
9	0	90

TABLE 3.5 *Nodal coordinates of the cross section in Figure 3.9*

Node m	Node n	ω_m (mm^2)	ω_n (mm^2)
6	5	-8.96	-82.66
4	4	-82.66	-233.4
4	3	-233.4	101.6
3	6	101.6	-8.96
8	7	-351.6	552.2
7	6	552.2	-8.96
3	2	101.6	248.3
2	8	248.3	-351.6
1	2	-581.3	248.3
8	9	-351.6	478.0

TABLE 3.6 *Numerical values of the principal warping function sketched in Figure 3.10*

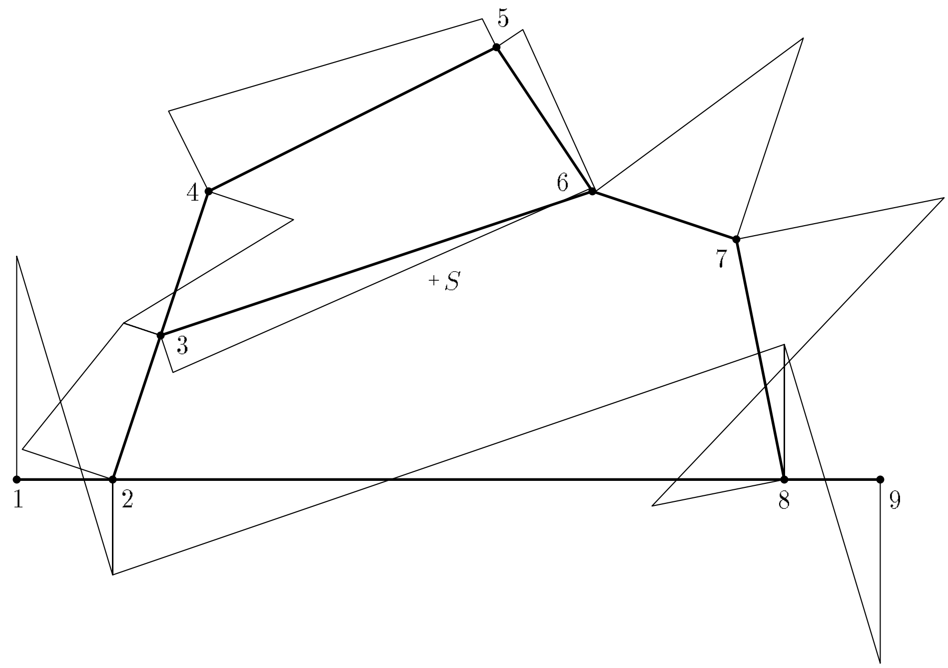

FIGURE 3.10 *Principal warping function for the section in Figure 3.9*

Thickness	Property	Linear Theory	BEAMSTRESS	Difference (%)
1 mm	Area (mm^2)	524	522	0.38
	I_y (mm^4)	455200	453976	0.27
	I_z (mm^4)	1143708	1136910	0.60
	I_{yz} (mm^4)	5270	4843	8.80
	J (mm^4)	749424	754965	0.73
	I_ω (mm^6)	$18.3\ 10^6$	$18.88\ 10^6$	3.07
5 mm	Area (mm^2)	2619	2571	1.87
	I_y (mm^4)	$2.276\ 10^6$	$2.25\ 10^6$	1.15
	I_z (mm^4)	$5.719\ 10^6$	$5.555\ 10^6$	2.95
	I_{yz} (mm^4)	26352	16740	57.40
	J (mm^4)	$3.747\ 10^6$	$3.892\ 10^6$	3.73
	I_ω (mm^6)	$91.5\ 10^6$	$116.1\ 10^6$	21.2
10 mm	Area (mm^2)	5237	5047	3.8
	I_y (mm^4)	$4.552\ 10^6$	$4.485\ 10^6$	1.5
	I_z (mm^4)	$11.44\ 10^6$	$10.81\ 10^6$	5.8
	I_{yz} (mm^4)	52703	19487	170
	J (mm^4)	$7.49\ 10^6$	$8.11\ 10^6$	7.6
	I_ω (mm^6)	$183\ 10^6$	$321\ 10^6$	43
12 mm	Area (mm^2)	6286	6010	4.6
	I_y (mm^4)	$5.46\ 10^6$	$5.38\ 10^6$	1.5
	I_z (mm^4)	$13.72\ 10^6$	$12.84\ 10^6$	6.9
	I_{yz} (mm^4)	63244	18361	244
	J (mm^4)	$8.99\ 10^6$	$9.90\ 10^6$	9.2
	I_ω (mm^6)	$219.6\ 10^6$	$437\ 10^6$	50

TABLE 3.7 *Properties of the cross section in Figure 3.9*